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GRAPHS:

OR THE GRAPHICAL REPRESENTATION OF ALGEBRAIC FUNCTIONS.

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The University Tutorial Series.

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GRAPHS:

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BY

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PREFACE.

THE main object of this book is the Graphical Treatment of the Variation of Algebraic Functions; but the subject is introduced by means of Statistical Graphs, as these constitute the most interesting and intelligible application of the Graphic Method.

The book contains numerous typical worked examples, with full explanations, and with squared paper diagrams inserted in the text. The diagrams are shown on tenth-inch squares, and the student is trained (i) to tabulate the values of the variables, (ii) to adapt his scales of representation to the requirements of these tabulated values, and (iii) to indicate the scales of representation on the diagram itself.

The book is well supplied with examples for practice, and numerical answers are given wherever required.

This book is primarily intended for students preparing for London Matriculation Examination. At the same time, for the sake of completeness, some matter clearly outside the present Matriculation Syllabus has been included; this additional matter is indicated by asterisks and may be

omitted if it is desired to make the course as short as possible.

The order of the chapters may in some cases be varied; thus, the early part of Chapter VII., on graphing simultaneous equations, may be taken after Chapter IV.

We have to thank Mr. A. G. Cracknell, M.A., for several valuable suggestions and for assistance with the proofs.

C. H. FRENCH.
G. OSBORN.

November 1903.

CONTENTS.

GRAPHS

OR THE

GRAPHICAL REPRESENTATION OF ALGEBRAIC FUNCTIONS

Sections and Examples marked with an asterisk should be omitted on a first reading.

CHAPTER I.

ON PLOTTING STATISTICS.

1. **Variables and Constants.**—A quantity which has not always the same value is called a **variable,** *e.g.* the population of London, the height of the barometer, the amount of light given by the moon, are variables.

A quantity which always has the same value is called a **constant,** *e.g.* the number of minutes in an hour, the length denoted by one inch, the amount of pure gold put into a sovereign, are constants.

2. **Connected variables: Definition of Graph.**—Two variables may be so connected that for every value of the one there exists some value of the other, *e.g.* at any instant of *time* a barometer has some particular *height.* The connexion may be shown by a diagram, which in the familiar case of the mercury barometer is given in various newspapers, and often exhibited in chemists' shops.

In Fig. 1, which is a copy of such a diagram, successive equal portions of time (days, here) are represented by equal lengths measured along the horizontal line AB; and the amount by which the height of the barometer exceeds 28 inches, at any instant, is represented by a vertical line

drawn upwards from the mark which corresponds to that instant. The upper ends of the vertical lines are joined by another line (an irregular one in this case) which shows how the barometer has varied during the whole time. This

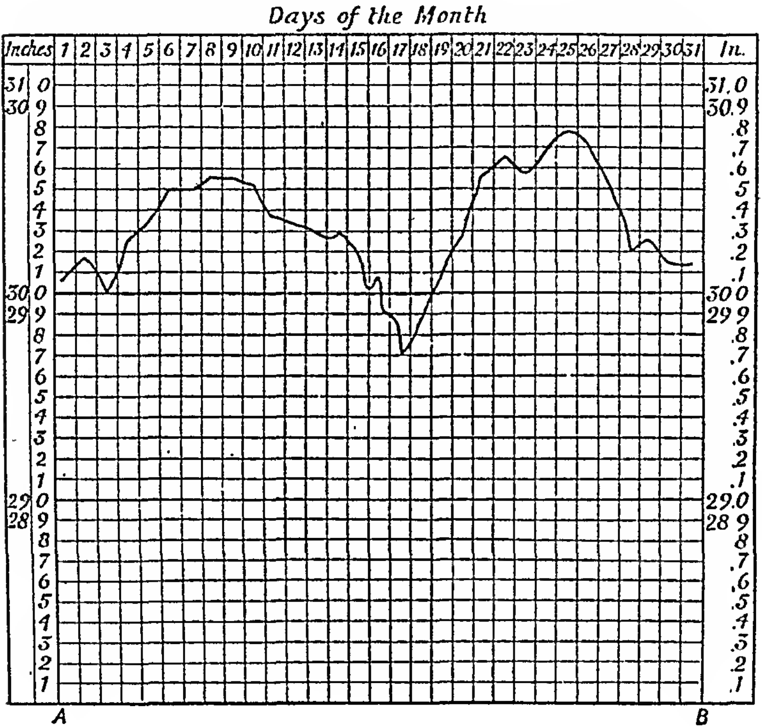

Fig. 1.

last line (taken in connexion with its *position on the diagram*) is called a **graph.**

DEFINITION.—A **graph** is a line drawn on a diagram so as to exhibit the nature of the relation between two connected variables.

3. **Examples of Graphs.**—Graphs of Statistics. Fig. 2 shows graphically how the population of London has increased; it is constructed from the subjoined table.

The figure is drawn on *squared paper*, ruled to tenths

Year.	1801	1811	1821	1831	1841	1851	1861	1871	1881	1891	1901
Pop. in millions.	0·9	1·1	1·4	1·7	1·9	2·4	2·8	3·3	3·8	4·2	4·5

of an inch, with every fifth line printed darker for rapid counting. To make it a convenient size, $\frac{1}{10}$th in. is taken as five years, on the horizontal line, and on the perpendiculars $\frac{1}{2}$ in. is taken as one million of population. The data are shown by the points actually marked—or "plotted," to use the customary word; these are then joined by a freehand

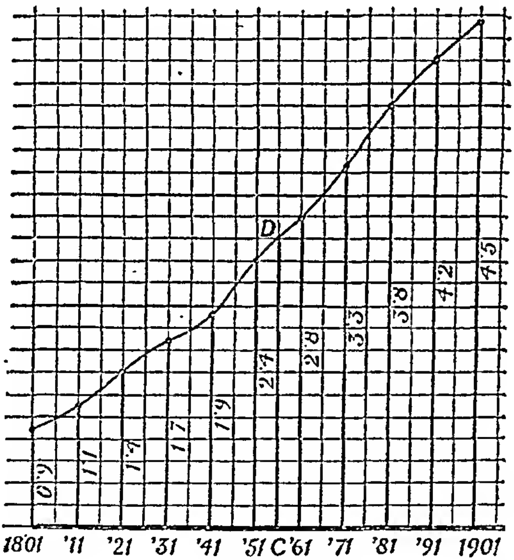

Fig. 2.

curve of as simple a nature and as free from irregularities as possible; and this curve (the graph) will supply information regarding intermediate points which were *not* plotted.

E.g. what was the probable population of London in 1856? The point C corresponds to 1856; draw the perp. CD. It represents, by inspection, 2·6 millions.

When was the population about $3\frac{1}{2}$ millions? (Ans. 1874).

These values, for points not plotted, are said to be found by **interpolation.**

5. **Advantages of Graphical Methods.**—There are certain very obvious advantages in recording statistics by means of a curve or graph.

(1) It is so easy to glance along the curve and see if it is regular; any peculiarities are noticeable.

(2) It is easy to find, by interpolation, values intermediate between those already plotted.

(3) It is sometimes possible to forecast the probable trend of a curve for a short distance.

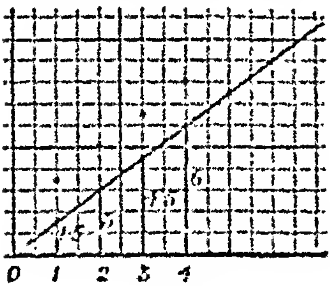

Fig. 3.

Such graphs may be used in business to register the rise or fall of prices, the cost of living per head, etc.; and sometimes useful information may be obtained by noticing how the price of some particular commodity rises or falls along with that of some other commodity. The information is presented in an attractive and manageable form. Statistics for graphical representation may be obtained from books like *Whitaker's Almanac,* or from the newspapers.

6. **Mathematical Graphs.**—In some cases a graph will turn out to be a straight line, or a circle, or some other recognizable curve.

Consider the case of water running from a tap; suppose that, in 1 min., 1·5 gallons have run out; in 2 min., 3 gallons; in 3 min., 4·5 gallons, and in 4 min., 6 gallons. Fig. 3 shows the plotted points, from which it is clear that the graph will be a *straight line.* The scales are $\frac{1}{4}$ in. for 1 min., $\frac{1}{10}$ in. for 1 gallon.

In future figures the perpendiculars will not be marked by thick lines.

As long as the supply continues, this graph will show how much water will have run out in *any* time—say 7·5 min. (11·25 gallons).

Suppose this water runs into a cistern which already contained 2 gallons; then if at each of the plotted points we produce the perpendicular upwards for a length 2, we plot 4 new points, showing the amounts of water in the cistern at the 4 given instants. These points, when joined by freehand, will show the graph for the water in the cistern at any other time; as before, the graph is a straight line.

The dots made when points are plotted should be *small* to show the positions accurately; they may be rendered more distinct by drawing a small circle round each.

Examples I.

Draw graphs for the statistics given in questions 1 to 10.

1. Approximate yearly consumption of sugar in the United Kingdom, in millions of cwts.

Year.	1883	1884	1885	1886	1887	1888	1889	1890	1891	1892
Amount.	22	22	24	21	24	23	25	25	27	26

(Take $\frac{1}{16}$ in. to represent 1 million cwt.)*

Mark points on the horizontal line for the *end* of each year.

2. Amounts of wheat imported into the United Kingdom in periods of 5 years (except the last).

5 years ending in	1862	1867	1872	1877	1882	1887	1892	1894
Millions of cwts.	41	34·6	42·1	54·3	64·2	55·8	61·9	70·1

(Take $\frac{1}{16}$ in. for 1 million, and estimate the decimals by eye.)

3. Average price of British wheat per quarter, in shillings and pence.

Year.	1890	1891	1892	1893	1894	1895	1896	1897	1898	1899	1900	1901	1902
Price.	31/2	35/5	33/1	26/8	23/5	21/5	24/10	28/8	36/2	26/0	26/4	27/1	28/4

* If millimetre paper is used, 2 (or 3) mm. might be taken instead of $\frac{1}{10}$ in.

4. Population of England and Wales in millions at the end of each year.

Year.	1891	1892	1893	1894	1895	1896	1897	1898	1899	1900	1901
Population.	29·0	29·4	29·7	30·1	30·4	30·7	31·1	31·4	31·7	32·3	32·5

Find the population in June 1895 and 1899, and March 1902.

5. Total yearly number of emigrants (including foreigners) from the United Kingdom, in thousands, from 1892 to 1901 inclusive—321, 308, 227, 272, 242, 213, 205, 241, 299, 303.
(Take 1 in. to represent 100,000 emigrants.)

6. The same as question 5, but referring only to emigrants of British origin—210, 209, 156, 185, 162, 146, 141, 146, 169, 172.
(Trace this graph on the same paper as question 5.)

7. Enrolled strength of Volunteers, in thousands, from 1890 to 1901 inclusive—221, 222, 225, 228, 231, 232, 236, 232, 231, 230, 278, 288. [Note any peculiarity in this graph, and assign a reason.]

8. The yearly imports of frozen beef, in millions of cwts., from 1892 to 1901 inclusive—2·08, 1·80, 2·10, 2·19, 2·66, 3·01, 3·10, 3·80, 4·13, 4·51. [Forecast the amount for 1902.]

9. The table shows the distances of certain railway stations from the terminus, and the times of departure of a train starting at 6.15.

Distance in miles.	0	$5\frac{1}{2}$	$11\frac{1}{2}$	$17\frac{1}{2}$	21	$24\frac{1}{2}$	28
Time. (hrs. min.)	6.15	6.28	6.39	6.51	6.59	7.8	7.17

(Measure times along OX. Scales, $\frac{1}{10}$ in. for 2 min., and for 1 mile.)
At which part of the journey is the greatest speed attained?

10. This is a similar example to No. 9; train starting at 5.25.

Time of departure.	5.25	5.27	5.30	5.32	5.35	5.42	5.44	5.48	5.51	5.57
Distance in miles.	0	$\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{3}{4}$	$2\frac{1}{4}$	4	$4\frac{1}{2}$	$5\frac{1}{4}$	$6\frac{1}{4}$	$7\frac{1}{4}$

(Allow 15 secs. stop before each departure.)
At which part of the journey is the greatest speed attained?

11. Water runs into a cistern, and the number of gallons in the cistern is noted at various times after it starts.

Minutes.	1	3	4	$4\frac{1}{2}$	$4\frac{3}{4}$	5
Gallons.	7	15	19	21	22	23

How much was in the cistern at first?

12. Water is running into a cistern, and during part of the time some of it is drawn off at a uniform rate by a tap. The table gives the amounts in the cistern after it has been running in for various numbers of minutes.

Minutes.	1	$1\frac{1}{2}$	$2\frac{1}{4}$	3	4	$4\frac{1}{2}$	$5\frac{1}{4}$
Gallons.	5	7	10	13	14	16	19

How much water is drawn from the cistern during this time, and at what rate?

CHAPTER II.

ALGEBRAICAL GRAPHS.

7. Graphs of Algebraic Functions.—Any expression containing a variable, x, and one or more constants, is called a function of x. Whatever the function is, it may be denoted by a single letter, say y.

When the value of x is changed, the value of y will also change. The connexion between y and x may be shown by a *graph*, any point of which is plotted by measuring the value of x along a *horizontal line*, and at the end of it drawing a line *vertically upwards* to represent the value of y.

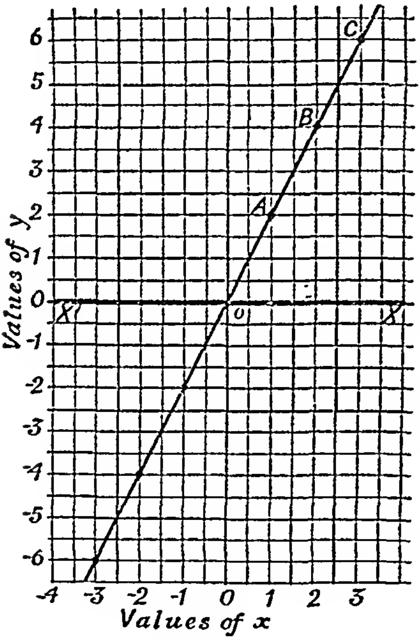

Fig. 4.

In Fig. 4, OX is the given horizontal line, drawn always *towards the right* from O which is the starting-point or **origin**.

8. *Ex.* 1.—Suppose we require the graph of the function $2x$ (so that in this case $y = 2x$). We choose simple values for x, such as 1, 2, . . ., and plot the corresponding points. We have taken $\frac{1}{6}$ in. to represent 1.

When x is	0	1	2	3
y is	0	2	4	6

These values of x and y give the points O, A, B, C in the accompanying figure. We can already see what kind of line would join all these points; it shows part of the graph of $2x$. It is also possible to consider *negative* values of x; it is obvious by Algebra that these would be measured in the reverse direction OX′.

When x is	-1	-2	-3
y is	-2	-4	-6

The values of y are now all *negative*, and are represented by drawing them in the reverse direction to the positive ones, viz. *below* XX′ instead of *above* it. See again Fig. 4.

Ex. 2.—As another example, consider the graph of $2x+3$; we will plot the points corresponding to $x = -3, -2, -1, 0, 1, 2$, as shown in the table.

x	-3	-2	-1	0	1	2
y	-3	-1	1	3	5	7

($\frac{1}{4}$ in. represents 1.)

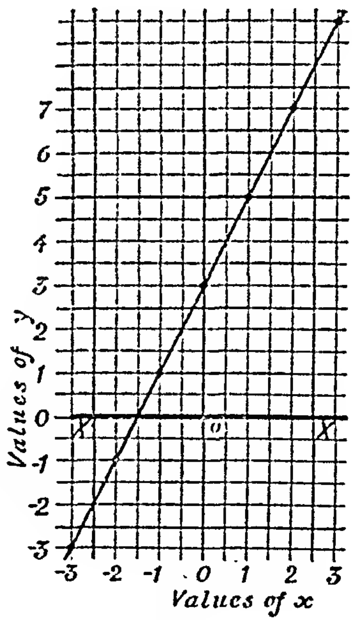

Fig. 5.

This turns out to be a straight line, and it illustrates what has just been said about negative values of y.

Ex. 3.—Using the graph of Ex. 2, find the values of y when $x = -\frac{1}{2}, \frac{1}{2}$, 1·5, 1·8; verify them from the equation. Mark the values of x on the horizontal line, and draw a perp. from each mark to meet the graph. We find by measurement that the lengths of the perps. are 2, 4, 6, and 6·6 respectively.

Ex. 4.—Consider now the graph of x^2, for which the equation will be $y = x^2$ (a very important case). We will plot the portion of this which extends from $x = -3$ to $x = +3$.

When $x = -3, -2, -1, 0, 1, 2, 3$;
$y = 9, 4, 1, 0, 1, 4, 9.$

We can now see what kind of curve (the thick line in Fig. 6) would join all these points; it shows us part of the graph of x^2. The shape of this curve should be carefully noticed; the origin divides it into two equal portions; also the value of y is never negative.

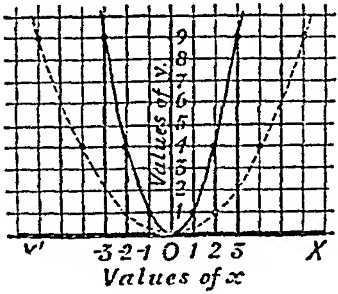

Fig. 6.

A graph of the same shape is shown on a larger scale in Fig. 13. The student is strongly advised to draw this graph for himself on a large scale, taking five divisions to represent 1 for both x and y.

Alteration of Scale.—The dotted line in Fig. 6 shows how the above graph changes when the size of the scale for x is doubled, while that for y is left unaltered. The graph has the same general character as before, but the plotted points are spread further apart.

Ex. 5.—Graph of $(x-1)(x-2)(x-3)$.

Let $y=(x-1)(x-2)(x-3)$, and form a table for fairly small values of x and y; thus

x	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
y	$-1\frac{7}{8}$	0	$\frac{3}{8}$	0	$-\frac{3}{8}$	0	$1\frac{7}{8}$	6

($\frac{4}{10}$ in. represents 1.)

It is clear from the equation that y cannot be 0 unless one of its three factors $x-1$, $x-2$, or $x-3$, is 0; this only happens when $x=1$, 2, or 3.

Fig. 7 shows the graph obtained by plotting these points, using a scale of 4 divisions to unity for the values of both x and y.

9. In plotting points, any value may be chosen at random for x (though numerically *large* values are obviously

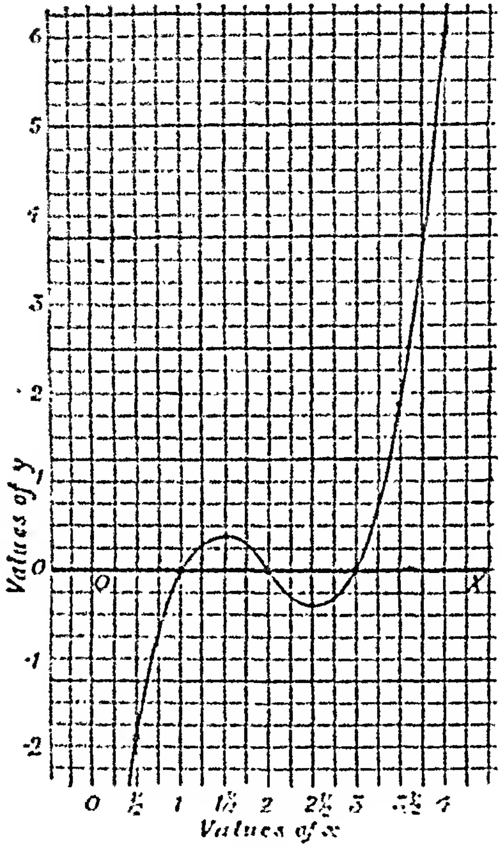

Fig. 7.

inconvenient); but the corresponding value of y depends on the value chosen for x: hence x is called the *independent variable*, and y is the *dependent variable.*

10. **Co-ordinates.**—In the accompanying figure (Fig. 8), let P be any point from which a line PM is drawn perpendicular to OX. Remembering the method given for plotting

points, it will obviously be a convenience to have *names* for the lines OM and MP, in connexion with the point P.

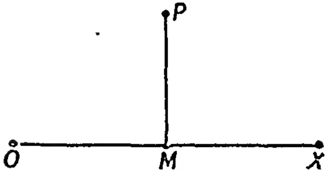

Fig. 8.

OM is called the *abscissa* of P,
MP is called the *ordinate* of P;

taken together, they are called the co-ordinates of P.

It is often convenient to draw through the origin O a line OY, perpendicular to OX, and then PN may be drawn from P perpendicular to OY (Fig. 9). Since PN = OM, we may take PN as the abscissa of P; the co-ordinates of P are then PN and PM. Again, since PM = ON, we may take ON as the ordinate of P, keeping OM for its abscissa; in this case P could be plotted by marking M and N, and then completing the parallelogram OP.

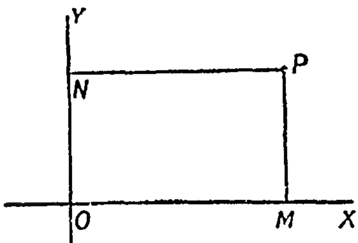

Fig. 9.

Sometimes OM is called the x-ordinate, or simply the x, of P; and then PM is called the y-ordinate, or the y, of P. The lines OX, OY, are called the **axes of co-ordinates.** Let a and b be given lengths; then the point for which OM = a and MP = b is called the point (a, b).

Ex. 1.—Plot the points (1, 2), (2, 5), (0, − 1), (− 1, − 4), ($\frac{1}{3}$, 0). (If correctly plotted, they will all be in a straight line.)

Ex. 2.—Plot the points (5, 5), (−6, −8), (−7, 4), (6, −7). (They form a square.)

Ex. 3.—Plot the points (5, 0), (4, 3), (3, 4), (0, 5), (− 3, 4), (− 4, − 3), (3, − 4). (They lie on a circle with its centre at the origin.)

We may remark that it is not always necessary to include the point (0, 0) in the figure. *E. g.* if in Ex. 3 we add 1000 to each number, the points to be plotted are (1005, 1000), (1004, 1003), (1003, 1004), (1000, 1005), etc. We cannot easily plot these to scale, but we can

show their relative positions by supposing that O represents the point (1000, 1000), and that OX, OY are merely lines *parallel* to the axes, not the axes themselves. (In Fig. 1, the *complete* height of the barometer is not shown on the graph.)

11. Straight Line Graphs.

(1) *Graph of ax*, where a is a constant. We will suppose that a is positive. See Fig. 10.

On OX take OM = 1, and measure MP = a, perpendicular to OX. Join OP and produce it. In OX take any point N, and draw NQ perpendicular to OX, meeting OP in Q.

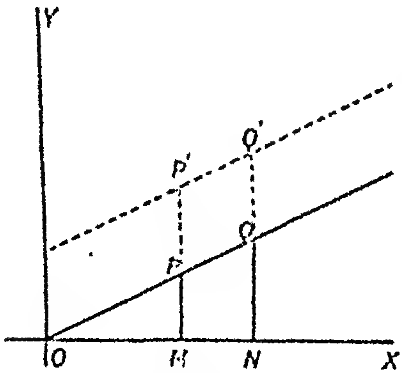

Fig. 10.

Then by similar triangles $\frac{QN}{ON} = \frac{PM}{OM} = \frac{a}{1}$.

$\therefore$ QN $= a$. ON $= ax$;

$\therefore$ Q is a point on the graph of ax. Similarly every point on OP (produced indefinitely both ways) is on the graph of ax, and *no other points* are on the graph (as shown easily by *reductio ad absurdum*).

(2) *Graph of $ax + b$.*—Here b is a constant as well as a. The graph is formed from the graph of ax, by producing the ordinates MP, NQ, etc., to P', Q', etc., making PP' = QQ' = etc. = b. (If b is *negative*, PP', QQ', etc., are measured *downwards* from P, Q, etc.)

Then P'M = PM + b, and Q'N = QN + b.

Hence P'M = a . OM + b,

and Q'N = a . ON + b.

Therefore both at P' and at Q' the ordinate = $ax + b$.

Thus P', Q', and in fact *all* points on the line P'Q', are on the graph of $ax + b$, and they are the *only* points on the graph.

We can, as usual, take y to represent the function we are dealing with (in this case $ax + b$), so that we may write

$$y = ax + b,$$

and we have just seen that the (unlimited) line P′Q′ contains all the points whose y and x satisfy this equation. Hence the line is often called the graph of the *equation* $y = ax + b$, instead of the graph of the function $ax + b$.

12. If we take any equation of the first degree in x and y, we can reduce it to the form $y = ax + b$. *E. g.* the equation $3x + 4y - 5 = 0$ can be reduced to the form $y = -\frac{3}{4}x + \frac{5}{4}$. It follows that every such equation expresses y as a function of x, and that the *graph* of that function is *a straight line.* This may be expressed briefly by saying that **every equation of the first degree in x and y represents a straight line.**

This statement is still correct, even if we are not using the same scale of representation for vertical quantities, as for horizontal. The student should prove this for himself.

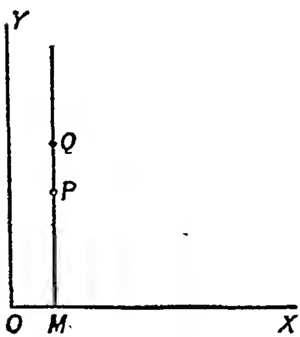

Fig. 11.

If the equation does not contain y, then it cannot be reduced to the above form; $x = 2$ is such an equation. The corresponding graph is found by plotting points for which the abscissa OM is 2, the ordinate having any value whatever, *e.g.* PM or QM (as in Fig. 11). Thus the graph is still a *straight line*, QPM, and it is parallel to the axis of y.

Note carefully that the equation $x = 2$ does not represent only one *point*, but a *straight line* of unlimited length (QPM), the abscissa of every point on this line being 2.

Any line perpendicular to the axis of x, like QPM, is exceptional, and is not likely to occur as a graph, because it does not allow different values to be taken for x (the independent variable).

On the other hand, a line *parallel* to the axis of x may easily occur as a graph; *e. g.* if a man 30 years old is 6 feet high, he will probably be still the same height at 40; the graph showing the connexion between his age and his height, taking values of x from 30 to 40, will be a horizontal straight line. This example shows *the graph of a constant.* "The graph of 6" means the graph of something which always has the value 6 whatever the value of x may be. The corresponding *equation* would be $y = 6$ (not $x = 6$).

*13. EQUATIONS REPRESENTING TWO STRAIGHT LINES.—If an equation of the *second degree* in x and y can be reduced to the form—

$$(ax + by + c)(a'x + b'y + c') = 0 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

(*i.e.* if the product of two linear factors is equal to zero), then it represents the *pair of straight lines* whose separate equations are

$$\left.\begin{aligned} ax + by + c &= 0 \\ a'x + b'y + c' &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

For if x, y are the co-ordinates of any point which satisfies *either* of the equations in (ii), they will cause the left side of (i) to become 0, *i. e.* they will satisfy (i). But if the co-ordinates of the point (x, y) do not satisfy *either* of the equations in (ii), then if these values of x, y are substituted in the left side of equation (i) it will be the product of two factors *neither of which is* 0, and $\therefore$ their product could not be 0. Hence no point which is *not* on one of the lines in (ii) can satisfy (i).

Ex.—Show that the equation

$$2x^2 + 5xy + 3y^2 - 5x - 7y + 2 = 0$$

represents a pair of straight lines, and draw the graph.

(Method: begin by putting the *terms of 2nd degree* into factors; then the *complete* factors, if there are any, will be obvious.) We have $2x^2 + 5xy + 3y^2 = (2x + 3y)(x + y)$; $\therefore$ by inspection of the remaining terms in the given equation, the factors are $2x + 3y - 1$ and $x + y - 2$ and the equations of the pair of lines are

$$2x + 3y - 1 = 0, \; x + y - 2 = 0. \quad \text{(See Fig. 12.)}$$

Note the easiest method for plotting the line represented by any equation of the first degree :—

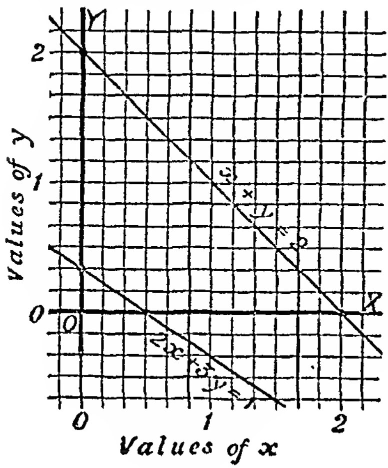

Fig. 12.

Take the first of these equations, viz. $2x + 3y - 1 = 0$. If $x = 0$, $y = \frac{1}{3}$; if $y = 0$, $x = \frac{1}{2}$; thus we have two points whose co-ordinates satisfy the equation, viz. $(0, \frac{1}{3})$ and $(\frac{1}{2}, 0)$. Now plot these points; the first obviously lies on OY and the second on OX. The straight line joining these points will be the required graph.

Similarly we can find the two points in which the line represented by $x + y - 2 = 0$ cuts the axes, viz. (0, 2) and (2, 0). (See Fig. 12.)

14. To find the equation to the straight line through two given points.

Ex. 1.—Find the equation to the straight line through the points (2, 4) and (3, − 1).

Let $y = ax + b$ be the required equation, where a and b are unknown constants. Since the point (2, 4) lies on the line, these values of the co-ordinates must satisfy the equation.

Thus $$4 = 2a + b \ldots \text{(i)}.$$

Similarly, since (3, − 1) lies on the line,
$$-1 = 3a + b \ldots \text{(ii)}.$$

Solving (i) and (ii) as simultaneous equations we find $a = -5$, $b = 14$. Hence the required equation is $y = -5x + 14$, or
$$5x + y = 14.$$

This method should not be used when the required line is *parallel to one of the axes,* a question which is decided by inspection of the co-ordinates.

Ex. 2.—Find the equation to the straight line joining (2, 4) and (2, − 1).

Since x has the same value for two points on this line, the line is parallel to the axis of y, and its equation is $x = 2$. (See § 12.)

15. On the choice of a scale of representation.— As we have already mentioned, this question is of great importance. The scales used along OX and OY need not be the same, and in each case the scale will depend on the magnitude of the values to be plotted.

Ex. 1.—Draw the graph represented by the equation

$$y = \tfrac{1}{1000}(x - 20)(x - 30)$$

between the values $x = 0$ and $x = 50$, and use it to determine the values of x which make y equal to ·3 and ·1 respectively.

We obtain the following table for Fig. 13.

x	0	5	10	15	20	25	30	35	40	45	50
y	·6	·375	·2	·075	0	– ·025	0	·075	·2	·375	·6

To obtain a good figure let us take one division of the paper to represent 2 in the values of x, and one division to represent ·02 in the values of y. We thus obtain the accompanying graph.

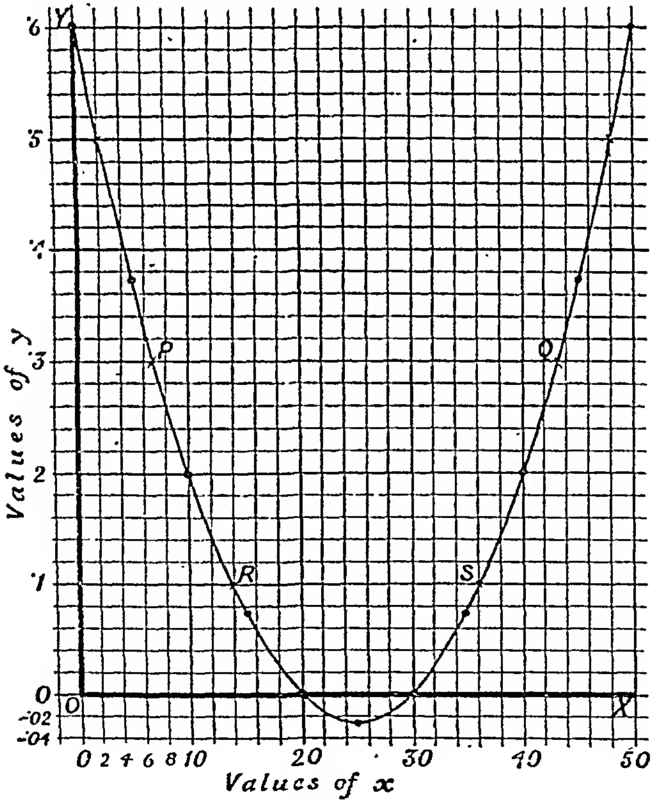

Fig. 13.

The value of y is ·3 at either of the points P and Q on the graph. The values of x at these points are about 7 and 43 respectively.

Similarly the points R and S show that y has the value ·1 when x is either 14 or 36.

Ex. 2.—Represent graphically the relation between the number of acres in a square field and the number of yards in its side, if the field is not larger than 70 acres. Use your graph to determine the length of the side when the area is (i) 20 acres, (ii) 40 acres, (iii) 60 acres.

It is easier to assume the length of the side and calculate the area than *vice versâ*. Let x represent the side of the field in yards, and y represent the corresponding area of the field in acres.

By a rough calculation we easily see that for the area to be 70 acres the length of the side must be a little under 600 yards. Thus we must plot for values of x from 0 to 600.

If the length of the side is x yards, the area of the field is x^2 square yards, or $\frac{x^2}{4840}$ acres. Thus $y = \frac{x^2}{4840}$.

We now construct our table.

x	0	50	100	150	200	250	300	400	500	600
y	0	·5	2·1	4·6	8·3	12·9	18·6	33·1	51·7	74·4

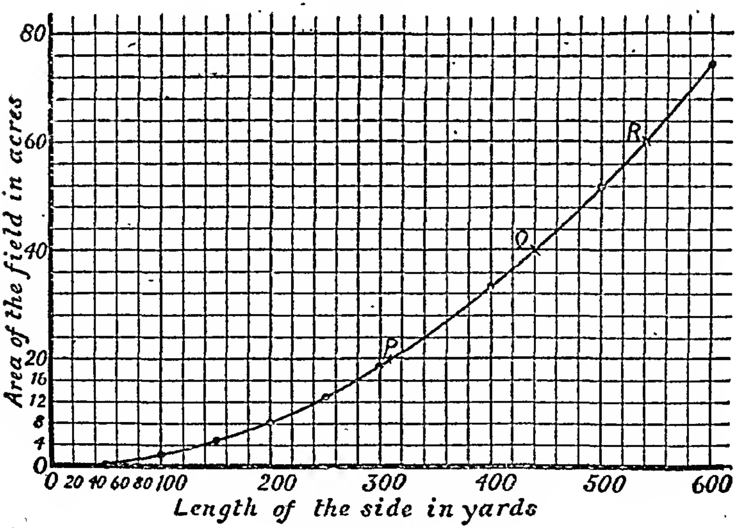

Fig. 14.

Taking scales of 20 yards to 1 division for x and 4 acres to 1 division for y, and plotting these values as accurately as we can, we obtain the accompanying graph.

The points P, Q, R correspond to areas 20, 40, 60 acres respectively. These points give the length of the side as 310, 440, 540 yards respectively. [The actual values are 311, 440, 539 respectively correct to the nearest yard.]

***16. Approximate Mathematical Graphs.**—If a number of points, when plotted, appear to lie approximately on some *mathematical graph*, it is of importance to be able to draw the simplest graph that agrees well with all the points, and to find its equation.

Ex. 1.—In Fig. 15 there are 9 points plotted in the space XOY, and they are roughly in a straight line. The straight line AB is drawn *by trial* so as to lie among them as evenly as possible, meeting the axes in A and B. (This is most easily done by stretching a black

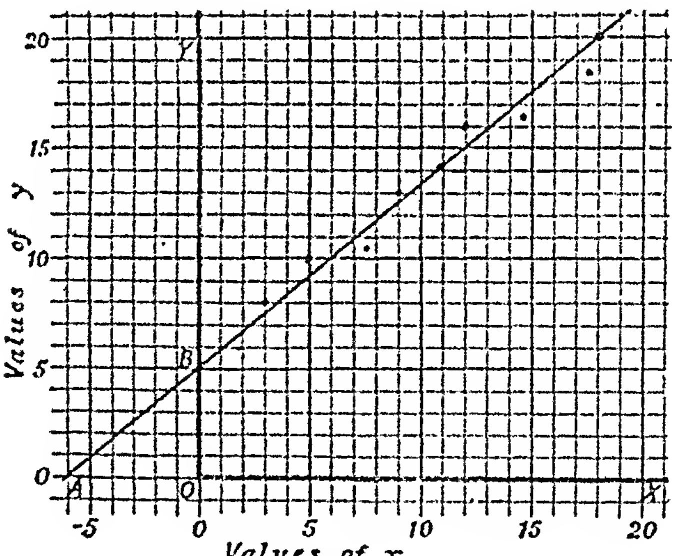

Fig. 15.

thread across the paper.) By inspection, A is the point $(-6, 0)$, and B is $(0, 5)$, whence we find that the equation of AB is $y = \frac{5}{6}x + 5$.

Ex. 2.—Show that the points (0, 0), (3, 2·5), (4, 4), (5·5, 7) lie approximately on a graph whose equation is of the form $y = ax^2$, and find a good value for a.

In Fig. 16 the 4 points are plotted ($\frac{2}{10}$ in. = 1). Now the *kind* of curve that must be drawn is obvious from § 8, *Ex.* 4; (for $y/a = x^2$ can be derived from $y = x^2$ by changing the scale for y). By drawing it exactly through the point (4, 4) we get $4 = a \times 4^2$, $\therefore a = \frac{1}{4}$, and the graph drawn is $y = \frac{1}{4}x^2$. By putting $x = 3$ or $5·5$ in this equation we find that the other given points are close to the graph and on opposite sides of the curve.

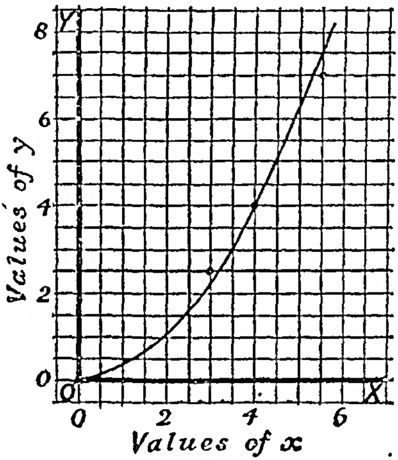

Fig. 16.

Examples II.

1. Draw graphs of the following functions of x, plotting at least 4 points for each.

(i) x. (ii) $3x$. (iii) $2x + 1$. (iv) $3x - 2$.

2. Draw graphs of

(i) $\frac{x}{2}$. (ii) $\frac{3x}{4}$. (iii) $\frac{x}{3} + 2$. (iv) $\frac{5x}{2} - 2$.

3. Draw graphs of

(i) $-x$. (ii) $-2x$. (iii) $-\frac{3x}{2} - 1$. (iv) $1 - \frac{3x}{2}$. (v) $\frac{x}{2} - \frac{1}{3}$.

(vi) $\frac{3}{4} - \frac{x}{2}$.

4. Draw graphs of the functions

(i) x^2. (ii) $\frac{x^2}{4}$. (iii) $-x^2$. (iv) $-\frac{x^2}{2}$. (v) x^2+1.
(vi) $(x+1)(x-1)$. (vii) $(x+2)(x-3)$.
(viii) $(x+2)(2x-1)$. (ix) $2x-x^2$. (x) $x-\frac{1}{2}x^2$.

5. Draw graphs of

(i) x^3. (ii) $-x^3$. (iii) $(x+1)(x-2)(x+3)$. (iv) $\frac{1}{2}x^2(x-1)$.
(v) $(x-1)^2(3-x)$. (vi) $\frac{x^4}{10}$.

6. Find the equations which represent the straight line graphs drawn through the following pairs of points:—[i] (0, 3), (2, 5); [ii] (5, 2), (3, 4); [iii] (– 2, 4), (10, – 2); [iv] (– 2, – 3), (8, – 4).

7. An inch is equal to 2·54 centimetres; draw a graph for expressing centimetres in inches, and use it to find the number of inches in 41 centimetres.

8. If C and F denote the readings on a Centigrade and Fahrenheit thermometer, corresponding to the same temperature, then $F = \frac{9}{5}C + 32$. Draw a graph for turning one scale into the other, and hence convert 60° F., 100° F., – 40° F. into the Centigrade scale.

9. A vessel of water was gradually heated, and the table shows its temperatures at various times, taken simultaneously on two thermometers A and B.

A	22·5	29	34·8	41	47	52	57	62	67·2
B	72	83	93	104	114·5	124	134	142	150·6

Show by a graph that there is a simple relation between the two scales, and find what it is.

Show also that some of the readings are incorrect.

10. The temperatures in question 9 were taken at intervals of half-a-minute. Draw a graph (using A) to show how the temperature varied with the time.

11. A spiral spring is fixed at its upper end, and has various weights hung in succession from its lower end so as to stretch it. From the following table draw a graph showing the relation between the weight

and the elongation it produces; and show that one of the observations is probably incorrect.

Weight (grammes)	0	200	400	600	800
Whole length of spring (cms.)	38·4	42·6	47·0	51·3	55·6

12. The time of swing of a simple pendulum depends only on its length. The table gives the lengths of certain pendulums, and the (approximate) number of beats they make in a minute.

Length in cms.	12	16	20	24	30	40	70	100
No. of beats per min.	172	148	135	123	110	95	72	60

Draw the graph, and hence find the number of beats per minute for pendulums of lengths 25, 35, and 50 cms.

13. From the table in question 12 find the time taken by each pendulum to make one beat, and graph the connexion between this and the square root of the length of the pendulum.

*14. Draw the graphs (in each case a pair of straight lines) corresponding to the equations—

(i) $x^2 - y^2 = 0$. (ii) $x^2 = 4y^2$. (iii) $y^2 = 2$.
(iv) $x^2 - 2xy - 3y^2 = 0$ (v) $x^2 - 2xy - 3y^2 - 3x - 5y + 2 = 0$.
(vi) $4x^2 + 4xy - 3y^2 - 2x - 3y = 0$.

15. Use the graph of $y = x^2$ to find the value of $(3{\cdot}5)^2$, $({\cdot}76)^2$, $\sqrt{7}$, $\sqrt{{\cdot}8}$.

16. When a body is falling freely under gravity the formula connecting the space described from rest (s feet), and the time taken (t secs.), is $s = 16t^2$. Draw the graph of this, and use it to find the space described in 4·5 seconds, and also the time taken to describe 25 ft.

17. Taking the formula $A = \frac{22}{7} r^2$ to represent the relation between the area of a circle and its radius, draw the graph, and find the area of a circle whose radius is 3 ft. 5 in.; also the radius of a circle whose area is 140 sq. in.

18. The area of the cross-section of a log of wood, at various distances from its thicker end, is given in the table (feet and square feet),

Distance	0	1	2	3	5	6
Area	3	2·25	1·8	1·5	1·125	1

Draw a graph showing the area of cross-section as a function of the length, and find the area at distance 4 ft.

*19. From the graph drawn for question 18, find the volume of the log. This is done by counting the number of squares included between the curve, the two end ordinates, and the axis of x. Each square represents 1 *cubic* foot.

If the log was equally thick everywhere, this would be obvious from the formula vol. = area of base × length. By imagining it divided into portions 1 ft. long, and counting the squares as above, the formula still applies to a close degree of approximation.

If in such a problem there are *many* incomplete squares in the figure, each of them should be reckoned as a whole square if it is greater than half a square, if not it should be omitted.

*20. Trace the graph in x and y given by the equations—

$$\begin{cases} x = t^2 + 4t + 3 \\ y = t^2 - 5t + 6 \end{cases}$$

from the values $t = 0$ to $t = 10$.

21. A manufacturer has priced certain lathes; the largest sells at £175 10*s.* and the smallest at £40. He wishes to increase his prices so that the largest sells at £200 and the smallest at £50. If the new price P and the old price Q are connected by the relation $Q = a + b\,P$, find the values of a and b to 1 decimal place, and find the new prices of lathes originally valued at £150, £125 10*s.*, £78,

CHAPTER III.

MAXIMA AND MINIMA.

17. Definition.—If a function of x increases in value while x is increased, and then begins to diminish when x is still *further* increased, the value of the function when the change occurs is called a **maximum**.

Fig. 17 shows the graph of $4x - x^2$. Putting $4x - x^2 = y$, it will be seen that while x increases from 0 to 2 there is a continual increase in y also; but if x is increased ever so little beyond 2 (*e. g.* take $x = 2{\cdot}001$), y begins to diminish. Thus $4x - x^2$ has a maximum value, viz. 4, when $x = 2$.

x	0	1	2	3	4
y	0	3	4	3	0

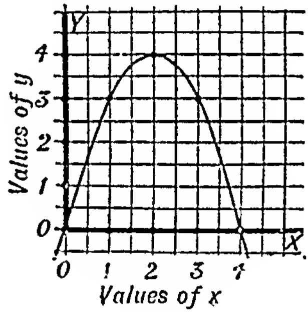

Fig. 17.

If a number of equidistant ordinates are drawn to the graph, close together, on each side of the maximum ordinate, it will be seen that the *difference in length* between two adjacent ones becomes smaller the nearer they are to the maximum; and when *very* near, it becomes small enough to be entirely negligible. Hence the length of the ordinate may be called **stationary**, when it is a maximum; at such a point, a very small alteration in x would produce *no* increase or decrease in y.

18. The meaning of a **minimum** value is now obvious.

Definition.—If a function of x decreases in value while x is increased, and then begins to increase when x is still

further increased, the value of the function when the change occurs is called a **minimum.**

Fig. 18 shows the graph of $x^2 - 4x + 3$. While x increases from 0 to 2, y diminishes from 3 to -1, and then begins to increase as soon as x is greater than 2. Hence this function has a minimum value, viz. -1, when $x = 2$.

x	0	1	2	3
y	3	0	-1	0

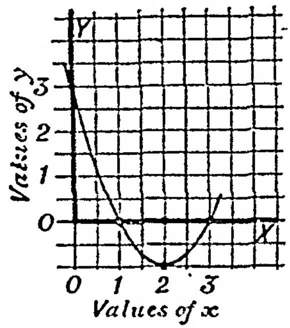

Fig. 18.

A minimum value is also a **stationary** value; this may be seen from Fig. 18, as in the case of a maximum value, by drawing a number of equidistant ordinates.

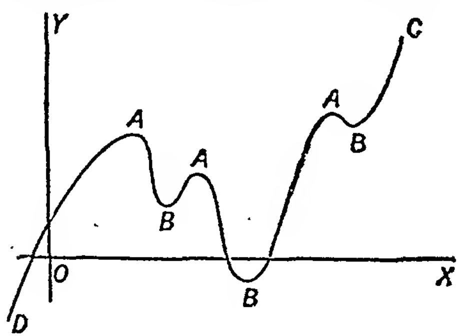

Fig. 19.

19. **Successive Maxima and Minima.**—If there are several maximum or minimum points, they follow one another *alternately;* Fig. 19, where maximum points are marked A and minimum B, will show that two A's do not occur without a B between. The graph may also go higher than any maximum, as at C, or lower than any minimum, as at D; also a minimum ordinate may be greater than a maximum.

20. Investigation of Maximum and Minimum Values of Functions.—Since the graph of any function of *the first degree* is a *straight line*, it is clear from the definition of maxima and minima that there are none on such a graph.

In other cases, the values of x, for which the function is a maximum or a minimum, can often be found by plotting a few points and drawing the graph.

Ex.—Find the maximum or minimum values of $x^3 - 3x$. Let $y = x^3 - 3x$, and form a table of points to be plotted; ($\frac{1}{2}$ inch $= 1$).

x	-2	$-1·5$	-1	$-0·5$	0	$·5$	1	$1·5$	2
y	-2	$1·12$	2	$1·37$	0	$-1·37$	-2	$-1·13$	2

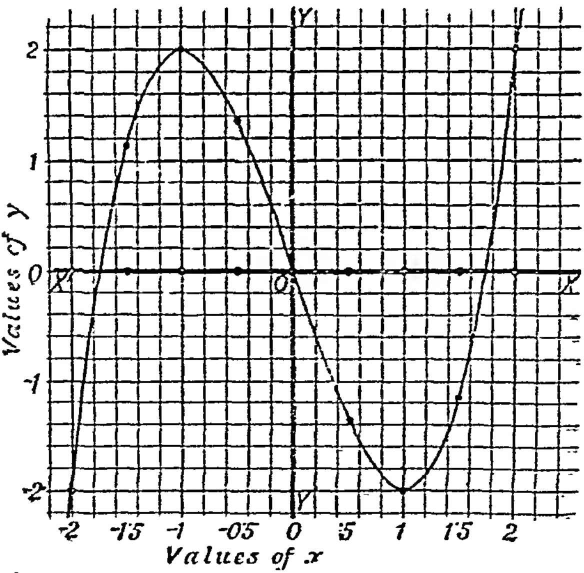

Fig. 20.

By examining these points (as plotted in Fig. 20) and the curve joining them, it will be inferred that y is a *maxi-*

mum, viz. 2, when $x = -1$, and a *minimum*, viz. -2, when $x = +1$. By plotting other points, *close to these two*, we can satisfy ourselves that these are correct; if they are not, the extra points enable us to find the proper positions.

For accurate work it is necessary to use a large scale of representation and to plot a large number of points.

*21. Sometimes it may not be easy to select points for plotting so as to show a maximum or minimum on the portion of the graph joining them. There is a mathematical process for discovering the maximum or minimum values (if any) of a given function, without drawing a graph. We have seen (Fig. 17) that if $y = 4x - x^2$, a maximum value of y occurs when $x = 2$. This could also be discovered as follows.

Let OM be the abscissa for which the ordinate PM is a maximum (see Fig. 21); denote OM by x_1, PM by y_1, and let $\text{ON} = x_1 + h$, $\text{OS} = x_1 - h$, where h (*i.e.* MN or MS) is a very small quantity; let $\text{QN} = y_2$, $\text{RS} = y_3$ (these are both *less* than y_1, if PM is a maximum).

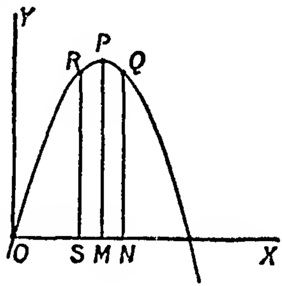

Fig. 21.

$$\begin{aligned} \text{Now } y_2 &= 4(x_1 + h) - (x_1 + h)^2 \\ &= 4x_1 - x_1^2 + 2h(2 - x_1) - h^2 \\ &= y_1 + 2h(2 - x_1) - h^2 \\ \therefore y_2 - y_1 &= 2h(2 - x_1) - h^2. \end{aligned}$$

Similarly, $y_3 - y_1 = -2h(2 - x_1) - h^2$.

But since y_1 is a *maximum*, $y_2 - y_1$ and $y_3 - y_1$ must *both* be negative, and this *will* be the case if $2 - x_1 = 0$, or $x_1 = 2$. If however x_1 was *not* equal to 2, they could not *both* be negative when h is taken very small, for the term h^2 being negligible compared with h, it is impossible for $2h(2 - x_1)$ to have the same sign as $-2h(2 - x_1)$.

This method is *general*, and applies equally to finding *minimum* values. It may be stated thus:—Let y be a function of x; then to find a value (x_1) of x for which the value (y_1) of y is a maximum or minimum, we change x_1 to $x_1 + h$, and let the corresponding value of y be y_2; express $y_2 - y_1$ in terms of x_1 and h; collect all the terms containing only the first power of h, and equate them to zero. This will give one or more values of x, for which y is either a maximum or minimum. To decide which it is, we can draw a portion of the graph in the neighbourhood of this value of x; or we may be able to decide by inspection whether the sign of $y_2 - y_1$ is positive or negative.

This method may fail if there are no terms containing h^2.

22. **Alternative Method.**—Another method for functions of the *2nd degree* will be sufficiently illustrated by an example. It depends on the fact that a square cannot be negative; *i.e.* the least value of n^2 is 0; for whether n be positive or negative, n^2 will be positive,

Ex.—If $y = x^2 - 6x + 11$, find a maximum or minimum value of y.

We have $y = (x - 3)^2 + 2$.

Hence a *minimum* value can be found, for $(x-3)^2$ will have its least value, viz. 0, when $x = 3$; and then $y = 2$, and is a minimum. If the given function had been $y = 2 - (x-3)^2$, this would have its *maximum* value when $x = 3$. In using this method, all the terms containing x must be included in the square.

Examples III.

1. Find graphically the maximum or minimum values of the following functions of x.

(i) $(x - 1)(x - 2)$. (ii) $(x + 1)(x + 2)$. (iii) $-x^2$.
(iv) $-(3x + 1)(x + 2)$. (v) x^4. (vi) $(x - 1)x(x + 1)(x + 2)$.

Verify these algebraically.

2. Find the maximum and minimum values of the following functions algebraically, and verify by drawing the graphs.

(i) $x^3 - 3x - 1$. (ii) $2x^3 - 3x^2 + 1$. (iii) $x^2 - x^3$. (iv) $x - x^3$.
(v) $\dfrac{x^2 - 8}{x + 3}$. (vi) $\dfrac{x^2 + 3}{x + 1}$.

3. The sum of two numbers being constant, find the change in their product as one of them increases from 0. Show that their product is greatest when they are equal, *e. g.* let their sum be 10, and draw the graph of $x(10 - x)$.

4. Find the maximum area of a rectangular field whose perimeter is 480 yards.

If one corner is at the origin, two of the sides being along the axes, draw the graph of the opposite corner.

5. The area of a rectangle is 9 square inches; find its minimum perimeter.

6. The area of a rectangular field is 20 acres; find the length of each side when the perimeter is a minimum.

7. What is the greatest possible area of a rectangle whose diagonal measures 10 inches?

8. What is the greatest possible perimeter of a rectangle whose diagonal measures 2 inches?

9. If $ab = 720$, find the minimum value of $3a + 5b$.

10. If $2a + b = 200$, find the maximum value of ab^2.

11. A rectangular cistern on a square base and without a lid is to be made, such that the total area of iron plating used in its construction is to be 100 sq. ft. If the side of the base measures x ft. determine the capacity of the cistern; and by means of a graph determine the maximum capacity which the cistern can have.

CHAPTER IV.

SOLVING EQUATIONS OF ONE UNKNOWN.

23. **First Method, by one Graph.**—The *method* to be followed is to bring all the terms of the equation to one side, so that the equation takes the form

$$(\text{some function of } x) = 0.$$

Then draw the graph of this function of x. Lastly find the values of x for the points in which the graph cuts the axis of x.

These values of x, thus found, will be the solutions of the original equation; *for at these points the values of y, i.e. of the given function of x, are zero.* The method will be understood quite easily from the examples which follow.

24. **Simple Equations.**—*Ex.*—Solve graphically the equation $4x=1$.

Write the equation in the form $4x-1=0$.

Draw the graph of $4x-1$, *i.e.* of $y=4x-1$, using the table—

x	-1	0	1
y	-5	-1	3

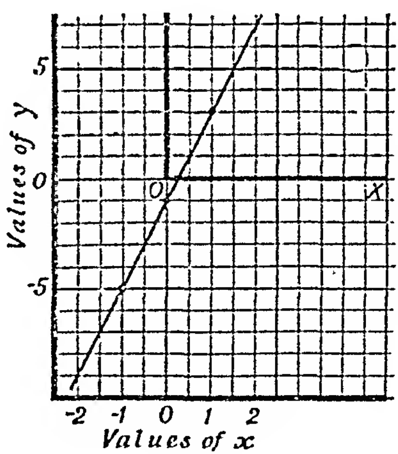

Fig. 22.

We see from Fig. 22 that the graph cuts OX at the point for which $x=\frac{1}{4}$. Observe that, for clearness, the x-ordinates are measured on double the scale of the y-ordinates.

This value of x is the solution of the equation.

25. **Quadratic Equations.**—*Ex.*—Find, graphically, the roots of the equation $x^2-3x=-2$.

Write the equation in the form $x^2-3x+2=0$.

Draw the graph of x^2-3x+2, using the table—

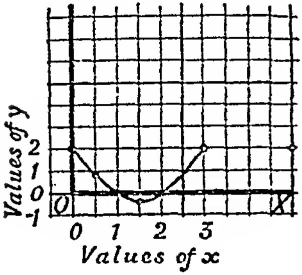

Fig. 23.

x	0	·5	1	1·5	2	3
y	2	·75	0	$-$·25	0	2

It will be seen from Fig. 23 that where the graph cuts OX the values of x are 1, 2. These are the roots of the equation.

26. The following example illustrates a method of solving a number of equations by means of one graph.

Ex.—Draw the graph of x^2-3x and use it to solve the equations (1) $x^2-3x-1=0$, (2) $x^2-3x+2{\cdot}25=0$, (3) $2x^2-6x+1=0$.

To draw the graph of x^2-3x write $y=x^2-3x$, and construct a table of values, thus:

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-3x$	6	3	0	-3	-6	-9	-12
y	10	4	0	-2	-2	0	4

The graph is shown in Fig. 24.

(1) To solve $x^2-3x-1=0$, write $x^2-3x=1$.

Then we must look on the graph of x^2-3x for the values of x, when $y=1$.

They are 3·3 and $-$·3, these are the roots of the equation, since these values of x make $x^2-3x=1$.

(2) To solve $x^2-3x+2{\cdot}25=0$, write $x^2-3x=-2{\cdot}25$.

Now find the values of x on the graph of x^2-3x when $y=-2{\cdot}25$.

There is only one value for x, viz. 1·5, this is the turning-point of the curve, and we can see that $x^2-3x+2{\cdot}25$ is the square of $x-1{\cdot}5$, so we should only expect to find one value for x.

(3) To solve $2x^2-6x+1=0$, write $x^2-3x=-\cdot5$, dividing by 2. Now find the values of x on the graph of x^2-3x, when $y=-\cdot5$. They are 2·82 and ·18 (or roughly 2·8 and ·2), these are the roots of the equation.

The solutions of these equations are marked on the graph, Fig. 24.

27. **When the roots are not small,** and hence the points at which the graph will cut the axis of x are not near the origin, it is often convenient to change the variable, in order to keep the table and the graph within bounds.

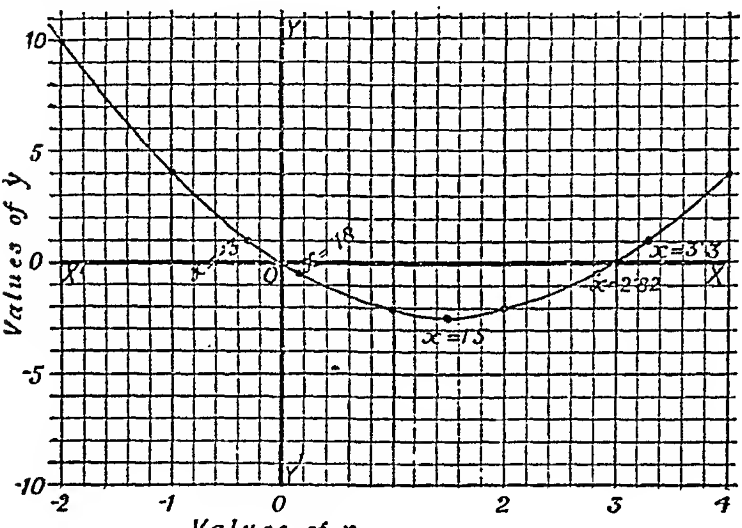

Fig. 24.

Ex.—Solve graphically the equation $x^2+11x-840=0$.

If we commenced forming a table by ascribing small values to x, the y-values would be numerically large, and it might not be easy to see what values to select for x in order to plot the part of the curve for which the y-values are small. By the following device the roots can be reduced to one-tenth of their original values.

Take some new letter, say z, equal in value to $x/10$, *i. e.* write $10z$ instead of x, in the original equation, giving

$$100z^2+110z-840=0,$$

or, dividing throughout by 100, the co-efficient of z^2,

$$z^2+1\cdot1z-8\cdot4=0.$$

Write $y=z^2+1\cdot1z-8\cdot4$, and draw the graph, using the table below where z is written for the reduced x-ordinate.

x	-4	-3	-2	-1	0	1	2	3
y	$3 \cdot 2$	$-2 \cdot 7$	$-6 \cdot 6$	$-8 \cdot 5$	$-8 \cdot 4$	$-6 \cdot 3$	$-2 \cdot 2$	$3 \cdot 9$

The student should draw the figure for himself. The curve cuts the axis of x at the points for which $z = 2 \cdot 4$, $-3 \cdot 5$, *i.e.* $x = 24$ or -35.

Clearly this is merely a method of reducing the scale of representation, and there is no necessity to limit ourselves to the particular fraction 1/10, chosen above, although this would be generally the most convenient.

28. **Equal Roots; Imaginary Roots.**—In the Theory of Quadratic Equations it is shown that an equation sometimes has equal roots or imaginary roots.

In the case of equal roots it will be found that the graph will touch the axis of x at one point only, or, to put it in another way, the graph will meet the axis of x in two coincident points. *E.g.* draw the graph of $x^2 - 4x + 4$. This will be found to touch the axis of x where $x = 2$, and this is the solution of the equation $x^2 - 4x + 4 = 0$.

In the case of imaginary roots the graph will not meet the axis of x at all, hence the graphical method gives no assistance in determining the magnitudes of such roots. If, when a graph is drawn, it is found that the graph does not meet the axis of x, the only conclusion to be arrived at is that there are no real roots. *E.g.* draw the graph of $x^2 + x + 1$. We see that the graph does not meet the axis of x, and hence we infer that the roots of the equation $x^2 + x + 1 = 0$ are imaginary.

29. **Equations of higher degrees than Quadratics.**—If such equations have real roots they may be found by the graphical method.

Ex.—Solve graphically the equation $x^3 + x^2 + x - 3 = 0$.
Forming a table for plotting points, we have

x	-2	-1	0	1	2
y	-9	-4	-3	0	11

Fig. 25 shows the shape of the curve, the x-ordinates are measured on a scale five times as large as the y-ordinates, the reason being that as some of the values of y are numerically large, it would be inconvenient to have a large scale for y, but if we have a small one, and also use it for x, the values of x will all be very close together on the graph. To avoid this we choose a large scale for x. It will be seen from the fig. that there is one real root, viz. $x=1$. The equation being of the third degree must have three roots, so the other two roots are imaginary. In this connection it is useful to remember that "imaginary roots occur in pairs," a theorem proved in Higher Algebra which must be taken for granted here.

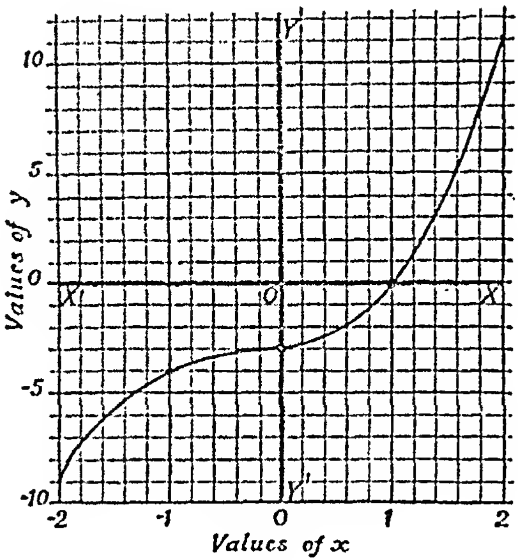

Fig. 25.

30. Second Method, by two Graphs.—The *method* to be followed is to arrange the equation so that the terms of the highest degree are on one side of the equation, and those of lower degrees on the other side. Then on one diagram draw the graphs of the separate expressions on each side. Measure the values of x at the points where these two graphs meet one another. These values of x will be the solutions of the original equation.

31. This method is not applicable to simple Equations, but it is the *best method for Quadratics.*

Ex.—Find to the first decimal place the roots of $x^2-3x-1=0$. Write the equation in the form $x^2=3x+1$.

Draw the graph of x^2, and also the graph of $3x+1$.

It will be found that they intersect at points whose x-ordinates are 3·3 and $-$ ·3, as in Fig. 26.

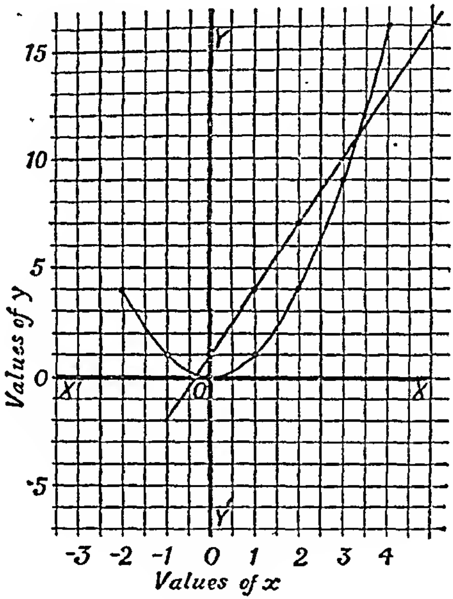

Fig. 26.

These values are the roots of the equation $x^2-3x-1=0$; for at either of these points the y-ordinates of the two graphs are identical, and hence the value of x^2 is equal to the value of $3x+1$.

Compare this method of solving the equation $x^2-3x=1$ with that in the first case of the last example in § 26.

32. In the case of **equal roots** the linear graph will *touch* the quadratic graph, instead of cutting it, and the value of x at the point of contact will be the solution of the equation.

In the case of **imaginary roots** the linear graph will not meet the quadratic graph at all.

These statements may be verified by trying to solve the equations given in § 28 by the two-graph method.

33. This method may be used for equations of higher degrees than quadratics, but it will be more troublesome than the first method unless one of the graphs is a straight line. In this latter case it is quite a satisfactory method.

Ex.—Solve the equation $x^3-3x-2=0$, by the intersection of two graphs. Write the equation in the form $x^3=3x+2$.

Draw the graphs of x^3 and of $3x+2$.

The tables for plotting will be

For x^3

x	-2	-1	0	1	2
y	-8	-1	0	1	8

For $3x+2$

x	-1	0	1
y	-1	2	5

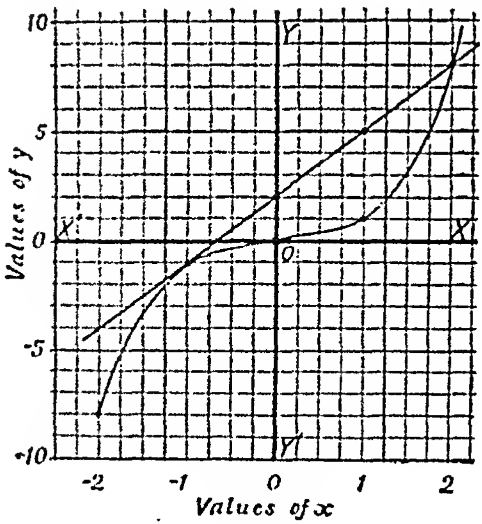

Fig. 27.

It will be seen from Fig. 27 that the line cuts the cubic curve at the point for which $x=2$, and touches it where $x=-1$. Hence the roots of the original equation are 2 and -1, the latter root -1 being really a pair of equal roots.

N.B. In drawing these graphs it is generally desirable to take a larger scale for x than for y, otherwise it will be difficult to see exactly where the linear graph cuts the quadratic graph.

Examples IV.

1. Draw the graph of the function $4x+1$, and hence solve the equation $4x+1=0$.

2. Draw the graph of the function $5x-7$, and hence solve the equation $5x-7=0$.

Find graphical solutions to the equations in questions 3 to 12, (1) by the method of one graph, (2) by the method of two graphs.

3. $x^2+3x+2=0$.
4. $x^2-x-30=0$.
5. $2x^2+3x-2=0$.
6. $3x^2+x-2=0$.
7. $x^2-6x+9=0$.
8. $4x^2+4x+1=0$.
9. $x^2-5x+7=0$.
10. $2x^2+x+\frac{1}{2}=0$.
11. $4x^2+x-1=0$.
12. $\frac{1}{2}x^2-x+\frac{1}{4}=0$.

Solve the equations in questions 13 and 14 by the method of § 27.

13. $x^2-5x-300=0$.
14. $x^2-33{\cdot}7x+283{\cdot}8=0$.

15. Draw the graph of x^2+4x, and use it to find the roots of
(1) $x^2+4x+3=0$, (2) $x^2+4x+4=0$, (3) $2x^2+8x-5=0$.

16. Draw the graph of $x^2-\frac{1}{2}x$, and use it to find the roots of
(1) $x^2-\frac{1}{2}x+2=0$, (2) $x^2-\frac{1}{2}x+1=0$, (3) $4x^2-2x=6$.

Find, correct to two places of decimals when possible, the roots of the equations in questions 17 to 24.

17. $x^3-1=0$.
18. $x^3+1=0$.
19. $x^3+2x-1=0$.
20. $x^3-3x+2=0$.
21. $x^3+3x^2+3x+1=0$.
22. $x^3+6x^2+11x+6=0$.
23. $2x^3-3x^2+1=0$.
24. $2x^3+3x^2-1=0$.

CHAPTER V.*

LIMITING VALUES. CURVES WITH ASYMPTOTES PARALLEL TO THE AXES.

34. **Meaning of Limiting Value.**—Consider the fraction $\frac{x+2}{x+3}$; if in this we put x successively equal to 5, 4, 3, 2, 1, 0 (a *decreasing* series), we shall get a *decreasing* series of values for the fraction, the largest being $\frac{7}{8}$ and the smallest $\frac{2}{3}$. Hence as x diminishes from 5 to 0, the fraction $(x+2)/(x+3)$ gradually decreases in value from $\frac{7}{8}$ towards $\frac{2}{3}$; and finally, when x is 0, the fraction reaches the value $\frac{2}{3}$. This may be expressed by saying that $\frac{2}{3}$ is the *limit* (or the *limiting value*) of the fraction $(x+2)/(x+3)$, when x diminishes to the value 0.

Again, consider the fraction $\frac{3}{x}$; if we put x successively equal to 100, 100^2, 100^3, 100^4, etc., the corresponding values of this fraction are $\frac{3}{100}$, $\frac{3}{100^2}$, $\frac{3}{100^3}$, $\frac{3}{100^4}$, etc. Thus the greater the value assigned to x the smaller is the corresponding value of the fraction; and if x is indefinitely increased the fraction $\frac{3}{x}$ becomes indefinitely small and ultimately reduces to zero.

Thus the limit of the fraction $\frac{3}{x}$, when x is indefinitely increased, is 0.

35. **Definition of Limiting Value.**—The Limiting Value of a function of an independent variable, for any particular value a of the variable, is that value from which the function differs by less than any assignable quantity, when the variable differs from a by less than any assignable quantity.

It has just been stated that the limit of $(x+2)/(x+3)$, when $x = 0$ is $\frac{2}{3}$. Now $\frac{x+2}{x+3} - \frac{2}{3}$ is $\frac{x}{3(x+3)}$, and when

x diminishes to 0, this difference $x/3\,(x+3)$ becomes less than any assignable quantity, and in fact vanishes.

36. *Limit notation.*—It is convenient to use the notation $\underset{x=a}{\mathrm{Lt}}$ with the meaning, "the limit when $x = a$ of." The symbol $\mathrm{Lt}_{x=a}$ is also used with the same meaning.

We have seen that $\underset{x=0}{\mathrm{Lt}}\dfrac{x+2}{x+3}$ is $\dfrac{2}{3}$, and $\underset{x=\infty}{\mathrm{Lt}}\dfrac{3}{x}$ is 0; as further illustrations observe that

$\underset{x=0}{\mathrm{Lt}}\dfrac{6x-4}{9x+8}$ is $-\dfrac{4}{8}$ or $-\dfrac{1}{2}$;

$\underset{x=\frac{2}{3}}{\mathrm{Lt}}\dfrac{6x-4}{9x+8}$ is 0, since the numerator vanishes, while the denominator is finite.

$\underset{x=-\frac{8}{9}}{\mathrm{Lt}}\dfrac{6x-4}{9x+8}$ is $-\infty$, *i.e.* it is an infinitely large negative quantity, since the denominator vanishes, while the numerator has a finite negative value.

37. **Undetermined Form $\frac{0}{0}$.**—It is not always possible to find, for a function of x, the value corresponding to a particular value of x, by merely substituting this particular value instead of x in the function. When this is the case the function is said to be undetermined, or indeterminate. A *fundamental instance* is $\underset{x=0}{\mathrm{Lt}}\dfrac{x}{x}$. This takes the form $\dfrac{0}{0}$, which cannot be interpreted as a fraction, for it has no numerator and no denominator.

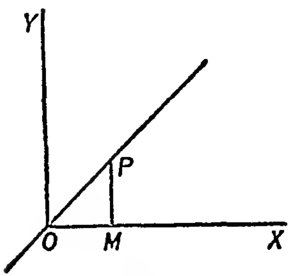

Fig. 28.

To understand this consider the graph of the line $y=x$, as in Fig. 28.

If from P, any point on the line, a perpendicular PM be drawn to the axis of x, the ratio PM : OM is always unity,

for $PM/OM = y/x = 1$. If P is taken *very near* to O, the ratio PM : OM remains unity (although both PM and OM are very small). Hence when x differs from 0 by less than any assignable quantity, the fraction $\frac{x}{x}$ differs from 1 by less than any assignable quantity (we write x in the numerator instead of y, since in this graph $y=x$ always); therefore *by the definition of a limiting value*, when $x=0$, the limit of $\frac{x}{x}$ is 1. The fact that $x/x=1$ *before* $x=0$ is immaterial.

It is clear that the same argument applies to a point at a very great distance from O along the line, viz. that the ratio $y:x$ still remains unity, even when both y and x are very large indeed. Hence $\underset{x=\infty}{\mathrm{Lt}} \frac{x}{x}$ is 1.

38. **Application to Infinite Values of x.**—To find the Limiting Value of $\frac{2x-1}{3x+4}$, when x is indefinitely increased, we may divide both numerator and denominator by x, thus reducing the fraction to the equivalent form $\dfrac{2-\frac{1}{x}}{3+\frac{4}{x}}$. It is permissible to divide by x, in this way, even when x is very large, since $\underset{x=\infty}{\mathrm{Lt}} \frac{x}{x}$ is 1 (end of § 37).

Now if x is indefinitely increased the fractions $\frac{1}{x}$ and $\frac{4}{x}$ are indefinitely diminished, and may therefore be neglected.

Thus $\underset{x=\infty}{\mathrm{Lt}} \left(2-\frac{1}{x}\right)$ is 2, and $\underset{x=\infty}{\mathrm{Lt}} \left(3+\frac{4}{x}\right)$ is 3.

$$\text{Hence } \underset{x=\infty}{\mathrm{Lt}} \frac{2x-1}{3x+4} = \underset{x=\infty}{\mathrm{Lt}} \frac{2-\frac{1}{x}}{3+\frac{4}{x}} = \frac{2}{3}.$$

From this example we see that the important numbers,

in the fraction $\frac{2x-1}{3x+4}$, when x is infinitely large, are 2 and 3, *i.e.* the co-efficients of the infinite quantity. It is immaterial what values the finite terms have.

E.g. $\underset{x=\infty}{\mathrm{Lt}} \frac{2x \pm a}{3x \pm b}$ is still $\frac{2}{3}$, whatever finite values we ascribe to a and b, and also when each or either of them is zero.

39. **Examples of Evaluation of Undetermined Forms.**

(1) $\underset{x=0}{\mathrm{Lt}} \frac{x^2-2x}{x^2-3x} = \underset{x=0}{\mathrm{Lt}} \frac{x(x-2)}{x(x-3)} = \underset{x=0}{\mathrm{Lt}} \frac{x-2}{x-3}$, removing $\frac{x}{x}$ since its limit when $x=0$ is 1 (§ 37), and $\underset{x=0}{\mathrm{Lt}} \frac{x-2}{x-3}$ is $\frac{-2}{-3}$ or $\frac{2}{3}$.

(2) $\underset{x=\infty}{\mathrm{Lt}} \frac{x^2-5x}{3x^2+4x} = \underset{x=\infty}{\mathrm{Lt}} \frac{2-\frac{5}{x}}{3+\frac{4}{x}}$, dividing numerator and denominator by x^2, remembering that $\frac{x^2}{x^2}$ is 1, even in the limit when $x=\infty$ (end of § 37), and $\underset{x=\infty}{\mathrm{Lt}} \frac{2-\frac{5}{x}}{3+\frac{4}{x}}$ is $\frac{2}{3}$, since, when $x=\infty$ both $\frac{5}{x}$ and $\frac{4}{x}$ are 0. This might also have been evaluated by observing that the important numbers are the co-efficients of the highest powers of x, when x is ∞, co-efficients of lower powers being immaterial, hence, at once, $\underset{x=\infty}{\mathrm{Lt}} \frac{2x^2-5x}{3x^2+4x}$ is $\frac{2}{3}$.

(3) $\underset{=\infty}{\mathrm{Lt}} \frac{x^3+2x^2+1}{3x^2+2x+1} = \underset{x=\infty}{\mathrm{Lt}} \frac{1+\frac{2}{x}+\frac{1}{x^3}}{\frac{3}{x}+\frac{2}{x^2}+\frac{1}{x^3}}$ (dividing numerator and denominator by x^3) $= \frac{1}{0} = \infty$, since $\frac{2}{x}, \frac{3}{x}, \frac{2}{x^2}, \frac{1}{x^3}$ all vanish when $x=\infty$. Or, otherwise, the highest power of x is x^3 which has 1 for co-efficient in the numerator, and 0 for co-efficient in the denominator (since there is no x^3 there), hence the limit is $\frac{1}{0}$ or ∞.

(4) $\underset{x=1}{\mathrm{Lt}} \frac{x^2-3x+2}{4x^2-x-3} = \underset{x=1}{\mathrm{Lt}} \frac{(x-1)(x-2)}{(x-1)(4x+3)} = \underset{x=1}{\mathrm{Lt}} \frac{x-2}{4x+3}$, removing the factor $\frac{x-1}{x-1}$, since its limiting value when $x=1$ is 1 (§ 37), and $\underset{x=1}{\mathrm{Lt}} \frac{x-2}{4x+3}$ is $-\frac{1}{7}$.

40. In the preceding paragraphs of this chapter an attempt has been made to give the student an idea of Limiting Values and how they may be obtained; we shall now show how these ideas may be applied to graphing functions in which zero and infinite values of the variables must be considered, in order to get the form of the curve, either near the origin, or at a very great distance from it.

41. **Example.**—Draw the graph of $y=\frac{1}{x}$.

On trying to form a table of values we see that for *positive values* of x, when x is very small y becomes very large, and as x approaches more and more nearly to zero y approaches more and more nearly to infinity; also when x is very large y becomes very small, and as x approaches more and more nearly to ∞ y becomes more and more nearly equal to zero. So for *negative values* as x approaches zero negatively, y approaches infinity negatively, and conversely.

We have the following table of values :—

x	$-\infty$	-5	-2	-1	$-\cdot5$	$-\cdot1$	-0	$+0$	$+\cdot1$	$+\cdot5$	$+1$	$+2$	$+5$	$+\infty$
y	-0	$-\cdot2$	$-\cdot5$	-1	-2	-10	$-\infty$	$+\infty$	$+10$	$+2$	$+1$	$+\cdot5$	$+\cdot2$	$+0$

Consider the positive half of the table: it is clear that as x gets larger the curve gets nearer and nearer to the axis of x, and although we cannot plot the point for which $x=\infty$ and $y=0$, we can indicate it as in Fig. 29 (see next page).

Definition.—A line to which a curve approaches more and more nearly till they are separated by less than any assignable distance, but without meeting it until an infinite distance is reached, is called an *asymptote*. (An asymptote may cut the curve at other points before the final approach begins.)

In Fig. 29 both axes are asymptotes to the curve. The curve consists of two parts, one part lying entirely in the first quadrant, as described above, and the other part, which can be deduced by a similar argument from the negative half of the table, lying entirely in the third quadrant.

42. Notice the method followed in order to plot the curve in the last paragraph. It is observed that $y=\infty$ when $x=0$, showing that the axis of y is an asymptote; also x and y are either both positive or both negative, this gives the shape of the curve for *very small values of x*. Next some points on the curve are plotted for *small finite values of x*, taking both positive and negative values. Lastly, when $x=\infty$, $y=0$, showing that the axis of x is an asymptote, this gives the shape of the curve for *very large values* of x. Thus it is possible to find a fairly accurate representation of the general shape of the curve.

43. It will be readily seen that the graph of any equation of the form $xy=a$, where a stands for some constant quantity, will be of a similar nature, viz. consisting of two parts, lying respectively in two opposite quadrants and having the axes of x and y for asymptotes.

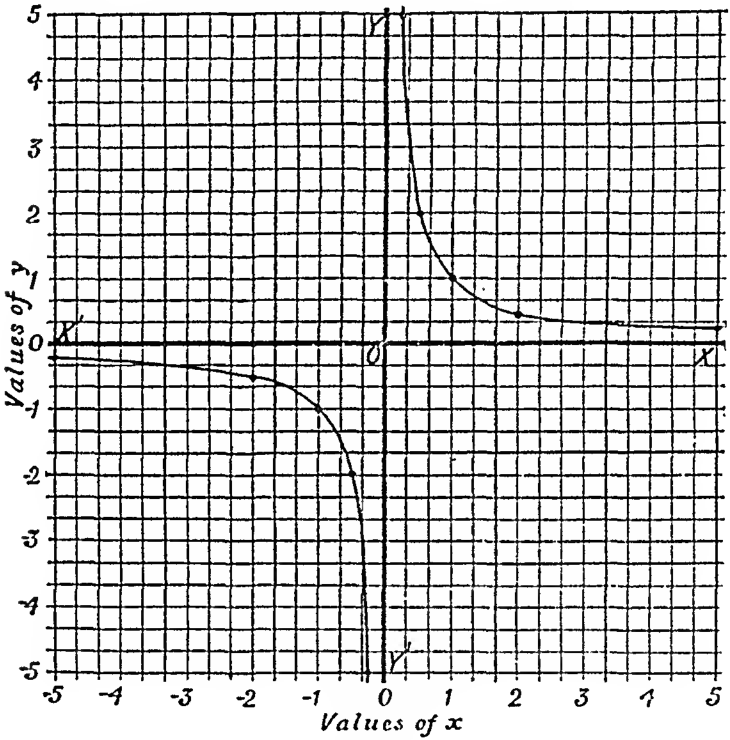

Fig. 29.

If a is positive the curve will lie in the first and third quadrants, while if a is negative the curve will lie in the second and fourth quadrants. Fig. 30, on the opposite page, shows the graph of $xy=-2$. The student should construct a table and confirm this.

44. **Curves with Asymptotes parallel to the Axes.—**

Example.—Draw the graph of $y=\dfrac{7-3x}{x-2}$.

First note the values of y when $x=\pm\infty$, viz. $y=-3$ (§ 38). Hence $y=-3$ is an asymptote. Next note that $x=2$ makes y infinite. Hence $x=2$ is also an asymptote.

Then find the points at which the curve cuts the axes of co-ordinates. Here when $x=\frac{7}{5}$, $y=0$ (this is the point at which the curve cuts the axis of x), and $x=0$ when $y=-3{\cdot}5$ (the point at which the curve cuts the axis of y).

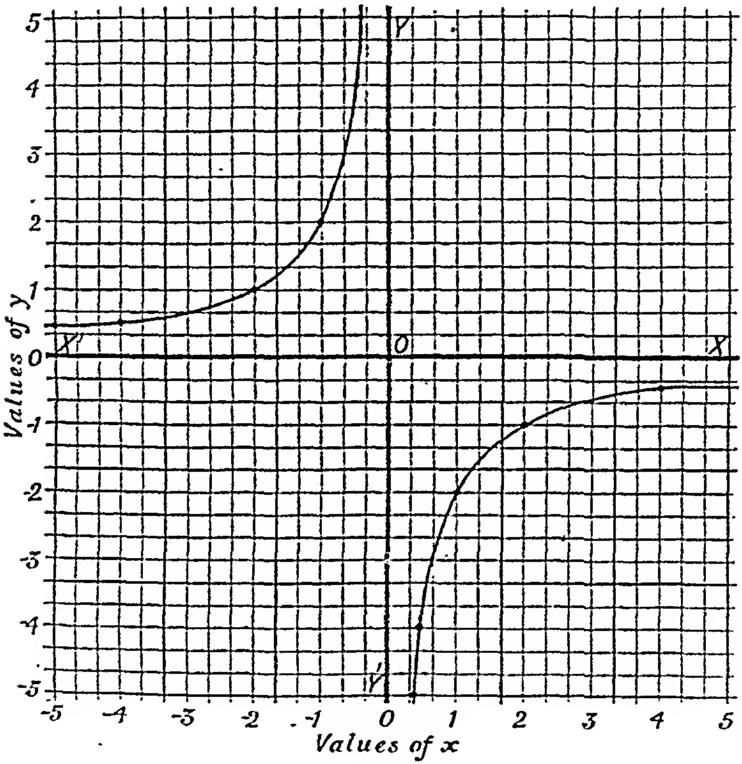

Fig. 30.

Lastly plot a few points for convenient values of x. To see what values of x are likely to be useful, *draw the asymptotes*. It will generally be desirable to plot a few points on either side of that asymptote which *is parallel to the axis of y*. In this example this asymptote is $x=2$, and we need to plot several points whose x-ordinates are a little greater than 2, and also several points whose x-ordinates are a little less than 2.

We give a table which will enable us to draw the graph with fair accuracy. If greater accuracy is desired more points must be plot[ted].

x	$-\infty$	-100	-5	0	1	1·5	1·99	2	2·01	2·1	2·33	3	4	$+100$	$+\infty$
$7-3x$		307	22	7	4	2·5	1·03	1	·97	·7	0	-2	-5	-293	
$x-2$		-10	-7	-2	-1	$-·5$	$-·01$	± 0	·01	·1	·33	1	2	98	
y	-3	3·01	$-3·1$	$-3·5$	-4	-5	-103	$\pm\infty$	97	7	0	-2	$-2·5$	$-2·99$	-3

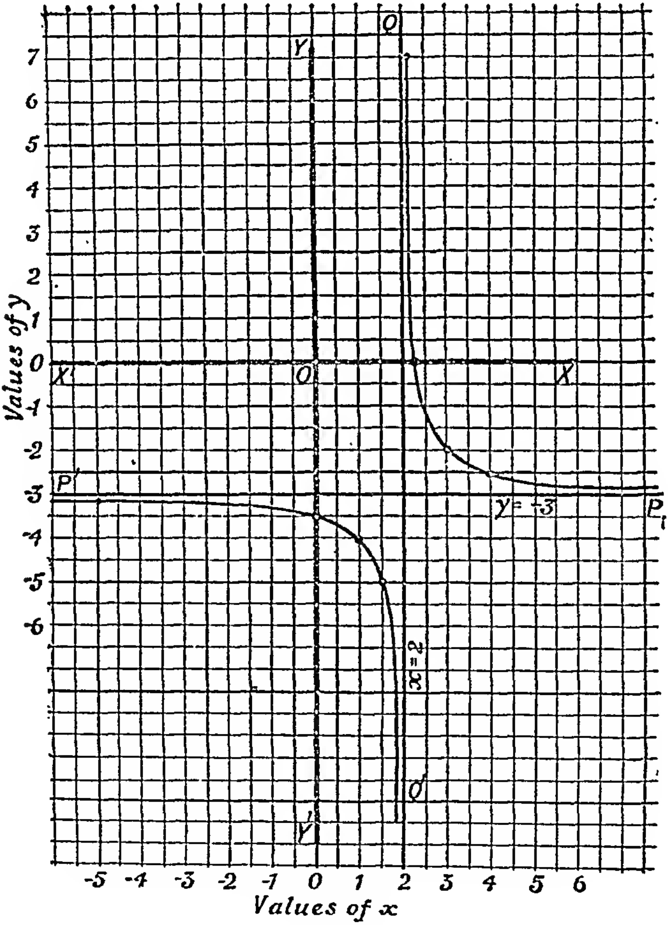

Fig. 31.

In the graph (Fig. 31) the axes are XX′, YY′, as usual, the asymptotes are PP′, QQ′.

Note particularly that when x is a very little less than 2, y is a large negative quantity, and when x is a very little greater than 2, y is a large positive quantity. Sometimes we express this by saying that when x passes through the value 2, y changes abruptly from $-\infty$ to $+\infty$.

45. It will not be difficult to see that $y=\dfrac{ax+b}{cx+d}$, where a, b, c, d stand for numbers, may be taken as the general type of such equations as that considered in the last paragraph, and that the resulting graph will have two infinite branches with a pair of asymptotes parallel to the axes. This type of curve is called a rectangular hyperbola. Some apparently more complex forms can often be treated in this way. *E.g.* draw the graph of $\dfrac{x^2-1}{x^2+3x+2}$.

Removing the factor $x+1$ from numerator and denominator we obtain $\dfrac{x-1}{x+2}$. The value of $\dfrac{x^2-1}{x^2+3x+2}$ will always be the same as the value of $\dfrac{x-1}{x+2}$, even when $x=-1$ (see § 39, Ex. 4).

The graph of $\dfrac{x-1}{x+2}$, or of $y=\dfrac{x-1}{x+2}$, may be drawn by the method of the preceding paragraph.

Examples V.*

Find the Limiting Values of the expressions in questions 1 to 6 (1) when $x=0$, (2) when $x=\infty$.

1. $\dfrac{x+1}{x-1}$. 2. $\dfrac{3x^2+x+2}{x^2+2}$. 3. $\dfrac{x^2+1}{x+1}$.

4. $\dfrac{2x+1}{2x^2+1}$. 5. $\dfrac{x(x-1)}{(x-2)(x-3)}$. 6. $\dfrac{(x+2)(x-1)}{x(x+1)}$.

Find the Limiting Values of the expressions in questions 7 to 10, for the values of x given.

7. $\underset{x=1}{\text{Lt}}\ \dfrac{x^2-1}{x+1}$. 8. $\underset{x=2}{\text{Lt}}\ \dfrac{x^2+2x}{x^2-2x}$.

9. $\underset{x=-1}{\text{Lt}}\ \dfrac{x^3-3x^2+3x-1}{x^3+3x^2+3x+1}$. 10. $\underset{x=-2}{\text{Lt}}\ \dfrac{x^3+8}{x^3+9}$.

Evaluate the undetermined forms in questions 11 to 14.

11. $\underset{x=0}{\mathrm{Lt}}\ \dfrac{3x^2+x}{2x^3+x}$.

12. $\underset{x=\infty}{\mathrm{Lt}}\ \dfrac{x^3}{4x^2}$.

13. $\underset{x=1}{\mathrm{Lt}}\ \dfrac{x^2-x}{x^2-3x+2}$.

14. $\underset{x=-2}{\mathrm{Lt}}\ \dfrac{x^3+8}{x^2-4}$.

Draw the graphs of the functions, or equations, in questions 15 to 32.

15. $xy=4$.

16. $xy=-4$.

17. $x+\dfrac{1}{x}$.

18. $x-\dfrac{1}{x}$.

19. $x+\dfrac{2}{x}$.

20. $x-\dfrac{2}{x}$.

21. $2x+\dfrac{1}{x}$.

22. $2x-\dfrac{1}{x}$.

23. $y=\dfrac{x}{x-2}$.

24. $y=\dfrac{3x}{x+1}$.

25. $y=\dfrac{2-x}{2+x}$.

26. $y=\dfrac{4x+3}{5x-6}$.

27. $y=\dfrac{2x-4}{2x+5}$.

28. $y=\dfrac{4-5x}{3-2x}$.

29. $y=\dfrac{2x^2-3x-2}{2x^2-5x+2}$.

30. $y=\dfrac{3x^2+x-2}{3x^2+5x+2}$.

31. $y=\dfrac{1}{x^2-1}$.

32. $y=\dfrac{x}{x^2-1}$.

CHAPTER VI.

GRAPHS SYMMETRICAL TO THE AXES.

46. It is easy to decide (as shown below) if an equation will lead to a *graph which will be symmetrical about* either axis of co-ordinates, *i.e.* if either axis divides the curve into two parts, such that one part could be folded over that axis so as to coincide exactly with the other part.

47. A graph will be **symmetrical about both axes** if all the powers of x and y which occur in the equation are even.

This will be understood if it is observed that when the co-ordinates (a, b) of any point satisfy the equation, the co-ordinates of each of three other points, viz. $(a, -b)$, $(-a, b)$, $(-a, -b)$, will also satisfy the equation, since whether we substitute a or $-a$ for x, and raise it to an even power, we get the same result, and so if b or $-b$ is substituted for y. Thus we shall get on the curve groups of points symmetrically situated with respect to either axis of co-ordinates, as shown in the next three figures.

Ex. 1.—Draw the graph of $x^2+y^2=4$.

Write the equation in the forms (1) $y=\pm\sqrt{4-x^2}$, (2) $x=\pm\sqrt{4-y^2}$. From (1) we can readily calculate the value of y corresponding to any assigned value of x, and we see that for each value of x there will be two values of y, numerically equal but of opposite signs. Hence for each point plotted above the axis of x there will be another point below the axis of x, at an equal distance from it, *i.e.* the axis of x will divide the curve symmetrically. So (2) enables us to calculate values of x corresponding to any assigned value of y, and it is clear that the axis of y divides the curve symmetrically. Also we see that the greatest possible numerical value for either x or y is 2, since a greater value would make the part under the *root-sign* negative.

Constructing tables for plotting points, we have,

From (1)

x	-2	-1	0	1	2
y	0	$\pm 1{\cdot}7$	± 2	$\pm 1{\cdot}7$	0

From (2)

y	-2	-1	0	1	2
x	0	$\pm 1{\cdot}7$	± 2	$\pm 1{\cdot}7$	0

The curve joining these points is a circle, see Fig. 32.

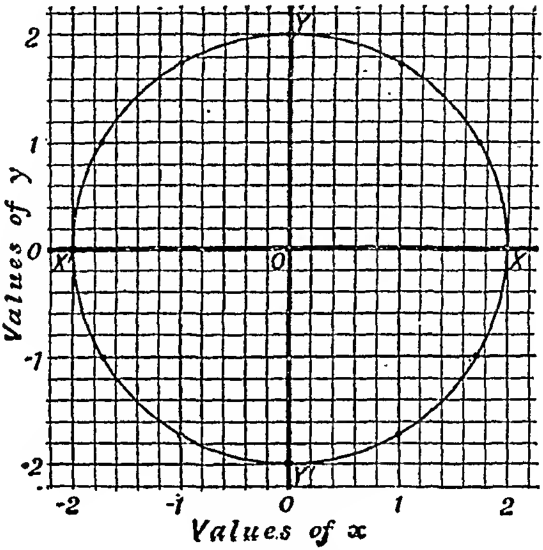

Fig. 32.

In fact the graph of the equation $x^2+y^2=c^2$, will always be a circle, radius c, provided that the vertical and horizontal scales of representation are equal.

Ex. 2.—Draw the graph of $4x^2+9y^2=36$.

Write this in the forms $y=\pm\frac{1}{3}\sqrt{36-4x^2}$ (1)

and $x=\pm\frac{1}{2}\sqrt{36-9y^2}$ (2).

From (1) we see that the greatest possible numerical value for x is 3, and from (2) that the greatest value for y is 2.

Construct a table for plotting points from (1), as below:

x	-3	-2	-1	0	1	2	3
y	0	$\pm 1{\cdot}5$	$\pm 1{\cdot}9$	± 2	$\pm 1{\cdot}9$	$\pm 1{\cdot}5$	0

Additional points can be plotted if desired, and a table constructed from the other form of the equation; but the points just plotted enable us to draw the curve with fair accuracy, as in Fig. 33.

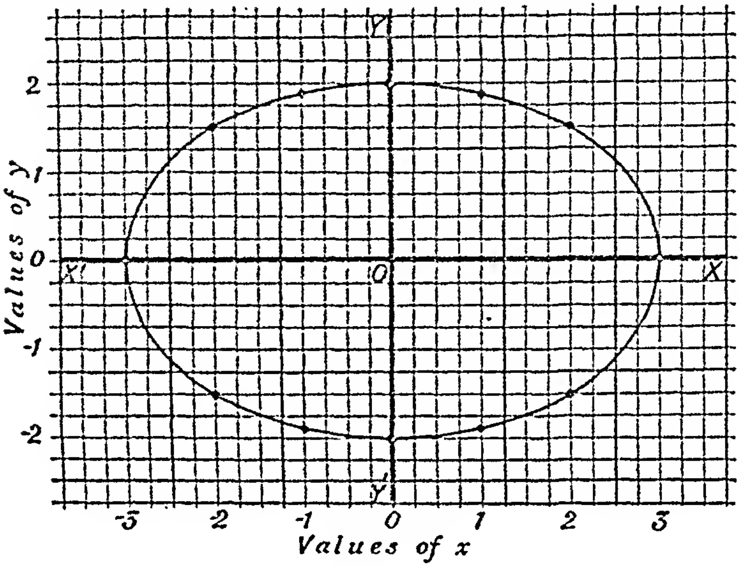

Fig. 33.

This class of curve is called an ellipse. Any equation of the type $ax^2+by^2=1$, in which the co-efficients of x^2 and y^2 are both positive but of different values will lead to an ellipse. The above equation is reduced to this type by dividing through by 36, when it becomes $\frac{1}{9}x^2+\frac{1}{4}y^2=1$.

**Ex.* 3.—Draw the graph of $4x^2-9y^2=36$.

Write this in the forms $y=\pm\frac{1}{3}\sqrt{4x^2-36}$ (1)

and $x=\pm\frac{1}{2}\sqrt{9y^2+36}$ (2)

It will be seen from (1) that x cannot be numerically less than 3, but there is no limit to the extent to which it may increase in magnitude, in fact x may increase numerically from ± 3 to $\pm\infty$, and y from 0 to $\pm\infty$. We shall attack this by the method of § 42, viz. plot a few points for small values of x and y and also try to indicate the form of the curve at an infinite distance.

In dealing with infinitely large values of x and y in any equation the *important terms* are always those of the *highest degree*, in this particular instance $4x^2-9y^2$, anything else may be disregarded (compare last part of § 38 and § 39, Ex. 2, 3); so the form of the curve $4x^2-9y^2=36$, *at infinity*, will be the same as that of the curve $4x^2-9y^2=0$.

Now $4x^2 - 9y^2 = 0$, or $(2x+3y)(2x-3y)=0$, consists of a pair of straight lines, viz. $2x+3y=0$, $2x-3y=0$ (see § 13).

As the curve approaches infinity, then, it approximates more and more nearly to these straight lines, and, in fact, these straight lines are asymptotes of the curve.

To construct the graph (1) draw the asymptotes $2x+3y=0$, $2x-3y=0$; (2) plot a few points on the curve for convenient values of x and y, remembering that x cannot be numerically less than 3, either positively or negatively; (3) draw a curve freehand through the plotted points and make it gradually approach the asymptotes.

x.	3	4	5	6
y	0	1·8	2·7	3·5

To draw the asymptotes observe that $2x - 3y = 0$ passes through the origin and also the point (3, 2), and $2x + 3y = 0$ passes through the origin and also the point (3, −2).

The table necessary to draw the curve is here given.

This gives four points in the first quadrant, the corresponding points in each of the other quadrants can be plotted by symmetry. The graph is shown in Fig. 34.

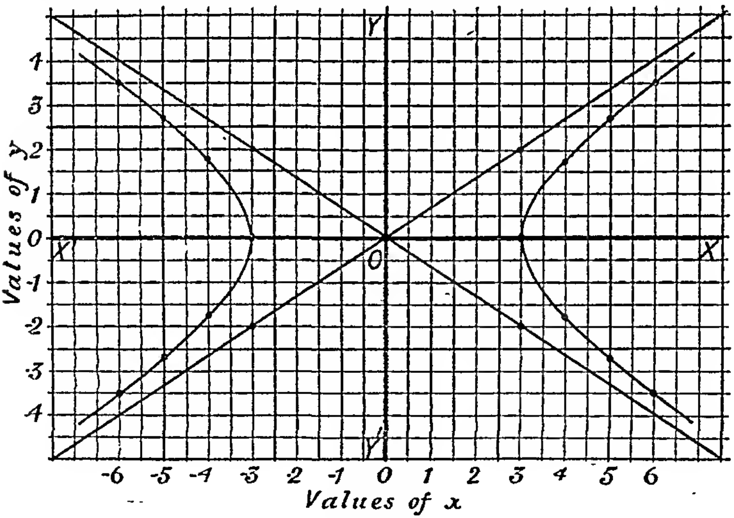

Fig. 34.

This class of curve is the hyperbola, not the rectangular hyperbola (which was shown in Chap. V), whose asymptotes are at right angles to one another. Any equation of the type $ax^2 - by^2 = 1$, in which the co-efficients of x^2 and y^2 are of opposite sign will lead to a hyperbola.

It should be noticed especially that $x^2 - y^2 = c^2$, where c stands for any number, is a hyperbola (not a circle); also since the asymptotes $x + y = 0$ and $x - y = 0$ are easily seen to be at right angles it will be a rectangular hyperbola.

48. **Graphs symmetrical about one axis only.**—A graph will be symmetrical with respect to the axis of x if the only powers of y present are even powers, and, similarly, a graph will be symmetrical with respect to the axis of y if the only powers of x present are even powers. This should be clear from the explanation at the beginning of § 47.

For instance, the graph of $y^2 = 2x + 4$ is symmetrical

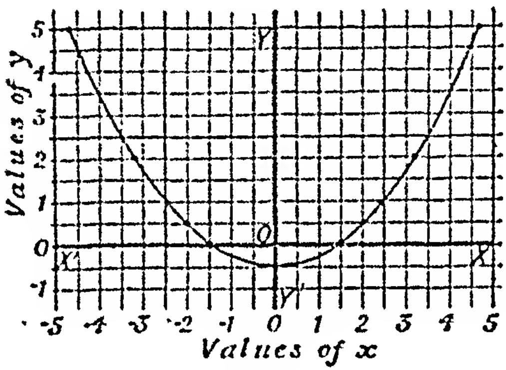

Fig. 35.

about the axis of x, while the graph of $x^2 = 4y + 2$ is symmetrical about the axis of y.

We will now draw the graph of $x^2 = 4y + 2$ (Fig. 35).

Notice first that $4y + 2$ must never be negative, hence the numerically greatest negative value possible for y is $-\frac{1}{2}$; but y can have any positive value. Now form a table so as to plot some points on the curve, and it will be best to assign values to y, and from these calculate the corresponding values for x, writing $x = \pm\sqrt{4y + 2}$.

y	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	5
$4y+2$	0	2	4	6	10	14	22
x	0	$\pm 1{\cdot}4$	± 2	$\pm 2{\cdot}4$	$\pm 3{\cdot}2$	$\pm 3{\cdot}7$	$\pm 4{\cdot}7$

The form of the curve is one which has been seen before, *e. g.* § 8, Ex. 4, § 25. The curve is called a *parabola*, and will be obtained as the graph of equations such as $y^2 = ax + b$, $x^2 = cy + d$, in which the second power of *one* of the variables only is present, the other variable being only of the first degree. It must not be inferred that this is the only form of equation by which a parabola may be represented, but it is the most common, and should be recognized. Notice that such an equation as $x^2 = -4y$ is included in the types given above. In this equation the values of y must always be negative, *i. e.* the curve must be entirely below the axis of x.

49. By inspection of the diagrams of the graphs drawn in § 47 it is clear that if the centre of the curve (in the hyperbola the point of intersection of the asymptotes is called the centre) were moved along the axis of x either way, the curve would remain symmetrical with respect to the x-axis of co-ordinates while not symmetrical with respect to the y-axis; so if the centre were moved along the y-axis the curve would be symmetrical with the axis of y only.

As an instance of a graph symmetrical with respect to the axis of y, draw the graph of $x^2 + y^2 - 2y = 0$.

We shall find after plotting a few points that it is a circle (compare the beginning of § 47).

Write the equation $x = \pm\sqrt{2y - y^2}$.

y cannot be >2, and y cannot have a negative value or else the part under the root sign would be negative giving imaginary values for x.

The following table is enough to draw the graph, showing that the centre of the circle (c) is at the point (0, 1), and the radius is 1.

y	0	1	2
x	0	± 1	0

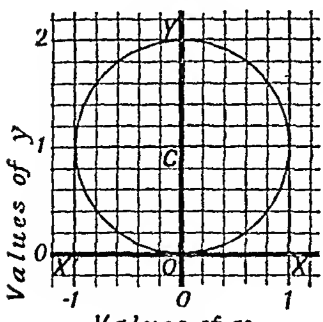

Fig. 36.

The graph is shown in Fig. 36.

Notice that this is an example of an equation with no constant term in it, so that the values $x = 0$, $y = 0$ satisfy it, *i. e.* the graph passes through the origin of co-ordinates.

50. A curve is said to be **symmetrical in opposite quadrants** if for every point (a, b) on the curve there is also a point on it whose co-ordinates are $(-a, -b)$. Such a curve need not be symmetrical about an axis.

This will be the case if the equation of the curve remains unchanged when the signs of both x and y are changed, as (for example) in the hyperbola $xy = 1$ (see Chap. V.), since if the point whose co-ordinates are (a, b) is on the curve, then the point $(-a, -b)$ will be on it; so also if the point $(a, -b)$ is on the curve, then the point $(-a, b)$ will be on it.

Ex.—Draw the graph of $x^3 - y = 12x$. Use the following table.

x	-4	-3	-2	-1	0	1	2	3	4
x^3	-64	-27	-8	-1	0	1	8	27	64
$-12x$	48	36	24	12	0	-12	-24	-36	-48
y	-16	9	16	11	0	-11	-16	-9	16

Also when $y = 0$, $x^3 - 12x = 0$; *i. e.* $x = 0$, or $x^2 - 12 = 0$, *i. e.* $x = \pm 2\sqrt{3} = \pm 3{\cdot}5$.

The graph is shown in Fig. 37.

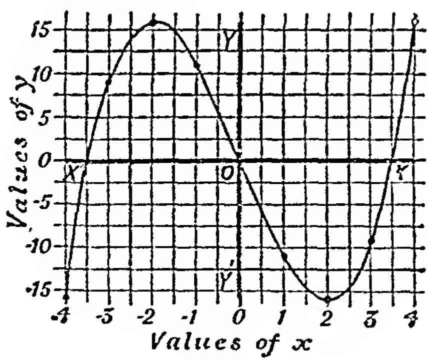

Fig. 37.

The correctness of the drawing can be tested by placing a ruler through the origin, then corresponding pairs of points in opposite quadrants should lie on it at equal distances.

EXAMPLES VI.

State whether the graphs of the equations in questions 1 to 6 are symmetrical about both axes, or about one axis (and if so, which), or in opposite quadrants (and if so, which).

1. $x^2=4y$.
2. $x^4+x^2y^2=4$.
3. $x^3-y^3=3xy$.
4. $y^2=8x-2$.
5. $x^2-y^3=3y$.
6. $x^2+y^2+2x+3=0$.

Draw the graphs of the equations given in questions 7 to 28. (The nature of the curve will be given in the answers, in most cases.)

7. $x^2+y^2=5$.
*8. $x^2-y^2=5$.
9. $2x^2+3y^2=5$.
*10. $2x^2-3y^2=5$.
11. $x^2+y^2-6x+8=0$.
12. $x^2+y^2+3y+1=0$.
13. $x^2+2y=4$.
14. $y^2+4x=4$.
15. $(x+y)^2=4$.
16. $(x-y)^2=8$.
17. $4x^2+y^2=1$.
18. $5x^2+6y^2=1$.
19. $x^3+y=2x$.
20. $x^3-y=2x$.
21. $x^3+y=2x+5$.
22. $x^3-y=2x+5$.
23. $y^2=(x-1)(x-2)(x-3)$.
*24. $y^3=(x-1)(x-2)(x-3)$.
*25. $y=\dfrac{(x-1)(x-3)}{x-4}$.
*26. $y=\dfrac{x(4-x)}{x+3}$.
27. $y^2=\dfrac{x^2(x^2-4)}{16}$.
28. $y^2=\dfrac{(4-x^2)(x^2-1)}{25}$.

Prove by geometrical argument that:—

29. The graph of the equation $x^2+y^2=c^2$ is a circle whose centre is the origin and whose radius is c.

30. The graph of the equation $ax^2+ay^2=b^2$ is a circle whose centre is the origin and whose radius is $b/\sqrt{a}$.

CHAPTER VII.

SOLUTION OF SIMULTANEOUS EQUATIONS.

51. To solve a pair of simultaneous equations graphically the method in every case is the following: draw the graph of each of the equations, taking the same origin, axes of co-ordinates and scale for each, then measure the co-ordinates of the point, or points, at which the graphs intersect. The co-ordinates of this point, or points, will be the solutions, the x-ordinate giving the value of x, and the y-ordinate the value of y.

The reason for this is that since any point at which the graphs intersect is on each graph, the co-ordinates of such a point satisfy the equation of each graph, hence these co-ordinates are solutions of the pair of simultaneous equations. Again, no point which is not on both graphs could satisfy both the equations. Hence all the real solutions are given by the co-ordinates of the point, or points, of intersection of the graphs.

It may happen that the equations given are inconsistent with one another, in that case the graphs will not intersect and there will be no common solutions to the equations. Also if equations have *imaginary* solutions, these will not be shown, since the graphs only mark real points.

52. Simultaneous Equations of the First Degree.

Ex.—Solve graphically the equations $2x+3y=6$, $4x-y=1$.

Draw the graph of $2x+3y=6$. This may be done most quickly by marking the points at which the line crosses the axes. Thus when $x=0$, $y=2$; and when $y=0$, $x=3$. These two points $(0, 2)$ and $(3, 0)$ are sufficient to determine the line. Then on the same figure draw the graph of $4x-y=1$. This passes through the points $(\cdot 25, 0)$ and $(0, -1)$.

In order to get an accurate result we have taken a large scale, and

we see (Fig. 38) that the graphs intersect at the point whose co-ordinates are ·64, 1·57. This point is on both lines and is the *only* point common, hence $x = \cdot 64$, $y = 1 \cdot 57$ are the solutions of the simultaneous equations.

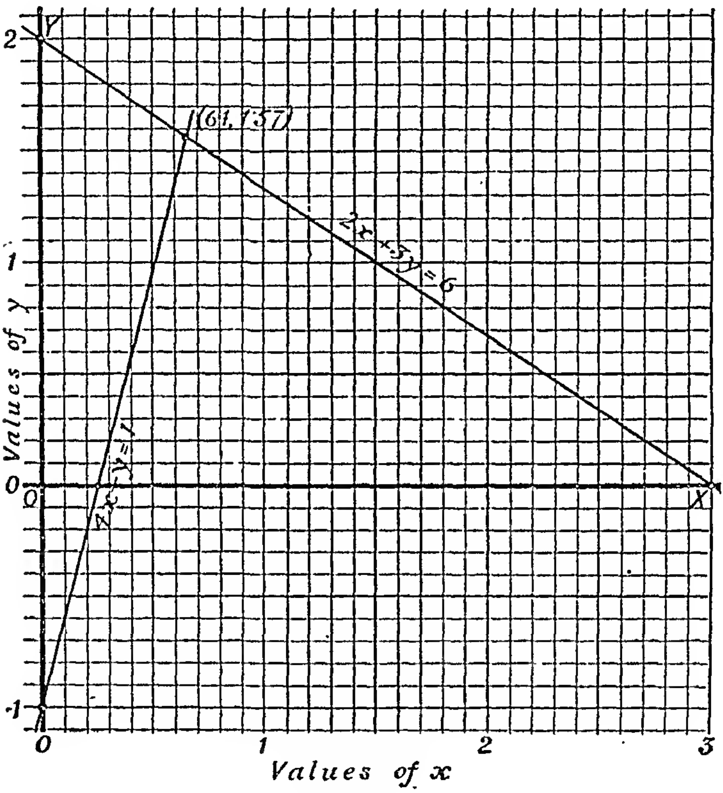

Fig. 38.

53. **Simultaneous Equations of Higher Degrees.**—Again the same method is to be adopted, viz. to draw the graphs and find the co-ordinates of the point, or points, of intersection; the only practical difficulty arising is to draw the graphs with sufficient accuracy to get the points of intersection correctly. Often a large scale must be adopted in order to show the point of intersection clearly.

We may have to do with a line (if one of the equations is of the first degree) and a curve, or two curves, and these may intersect in several points, or touch one another, or may not meet at all.

The most useful case is that in which by the graphical method we are enabled to solve equations whose roots are difficult to find by ordinary algebraical methods.

Ex. 1.—Find approximately the roots of the simultaneous equations $x+y=2$, $x^3-2y=1$.

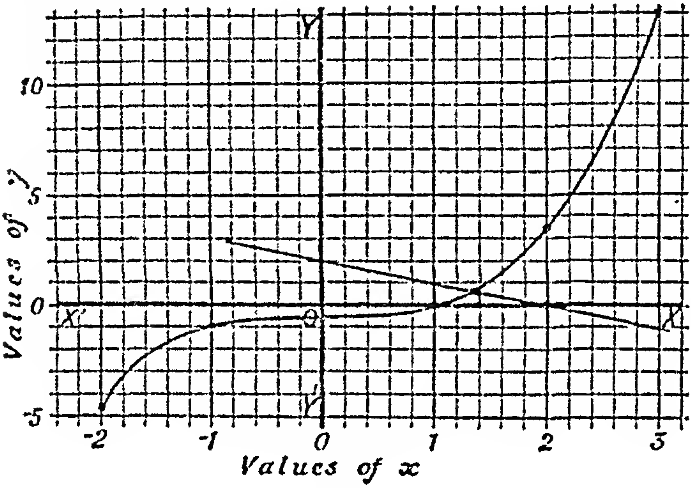

Fig. 39.

For the line $x+y=2$, observe that it passes through the points (2, 0) and (0, 2). Use the table below for the graph of $x^3-2y=1$. After plotting a few points it is clear that not much need be drawn:

x	-2	-1	0	1	2	3
x^3-1	-9	-2	-1	0	7	26
y	$-4\cdot5$	-1	$-\cdot5$	0	$3\cdot5$	13

and notice that as x increases negatively so does y, and as x increases positively y increases positively.

The graphs are shown in Fig. 39, the scale for x being taken five times as large as that for y, since y increases in value much more rapidly than x. It is difficult to get the co-ordinates of the

point of intersection very accurately, the values obtained in the figure are $x=1{\cdot}35$, $y={\cdot}65$.

Ex. 2.—Solve graphically the simultaneous equations $x^2+y^2=12$, $x^2+2y=4$.

The graph of $x^2+y^2=12$ is a circle, centre at the origin and radius $\sqrt{12}$, say $3{\cdot}46$. This should be obvious from Chapter VI., but even if not, a table can be constructed as in § 46 and the curve drawn from the table The graph of $x^2+2y=4$ is a parabola symmetrical about the axis of y (see Chapter VI., § 48).

x	0	± 1	± 2	± 3
$4-x^2$	4	3	0	-5
y	2	$1{\cdot}5$	0	$-2{\cdot}5$

Since the equation may be written $x^2=4-2y$, it is clear that the greatest possible *positive* value of y is 2; but y may have any less positive value or any negative value. To form a table of values write the equation in the form $y=\dfrac{4-x^2}{2}$.

On drawing the graphs as in Fig. 40 we see that they intersect at the points $(2{\cdot}8, -2)$ and $(-2{\cdot}8, -2)$.

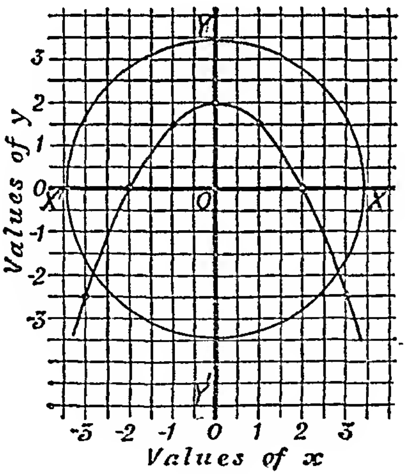

Fig. 40.

Hence the solutions are $x=2{\cdot}8, -2{\cdot}8$
$y=-2, -2.$

If we solved these equations algebraically, we should also obtain two imaginary values for x when $y=4$. These solutions are not shown by the graphs (see § 51).

EXAMPLES VII.

Solve the following equations graphically, and verify your solutions algebraically.

1. $\left.\begin{array}{l} y=3x-1 \\ y=x+1 \end{array}\right\}$

2. $\left.\begin{array}{l} y=9-x \\ y=2x-6 \end{array}\right\}$

3. $\left.\begin{array}{l} 3x+2y=8 \\ 2x-3y=1 \end{array}\right\}$

4. $\left.\begin{array}{l} 3x-5y=0 \\ 3x+5y=30 \end{array}\right\}$

5. $\left.\begin{array}{l} 2x+5y=8 \\ 5x-2y=6 \end{array}\right\}$

6. $\left.\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y=4 \\ \frac{1}{4}x+y=1 \end{array}\right\}$

7. $\left.\begin{array}{l} 2x+y=4 \\ x+\frac{1}{2}y=3 \end{array}\right\}$

8. $\left.\begin{array}{l} 3x-y=2 \\ \frac{2}{3}x+\frac{1}{3}y=x \end{array}\right\}$

9. $\left.\begin{array}{l} y=x^2+1 \\ x+y=2 \end{array}\right\}$

10. $\left.\begin{array}{l} x^2+2y=1 \\ 4y-x-2=0 \end{array}\right\}$

11. $\left.\begin{array}{l} x^2+y^2=8 \\ x-y=2 \end{array}\right\}$

12. $\left.\begin{array}{l} x^2+y^2=5 \\ y-2x=5 \end{array}\right\}$

*13. $\left.\begin{array}{l} x^2-y^2=2 \\ 2x+y=5 \end{array}\right\}$

*14. $\left.\begin{array}{l} x^2-4y^2=4 \\ 3x+5y=8 \end{array}\right\}$

15. $\left.\begin{array}{l} x^2+y^2=16 \\ xy=6 \end{array}\right\}$

16. $\left.\begin{array}{l} x^2+y^2-4x=8\cdot 6 \\ xy=-3 \end{array}\right\}$

17. $\left.\begin{array}{l} x^3+2y=2 \\ 2y+3x=4 \end{array}\right\}$

18. $\left.\begin{array}{l} y^3-2x=3 \\ 3y-x=6 \end{array}\right\}$

*19. $\left.\begin{array}{l} y^3+4y^2-2x=0 \\ 2x=5y \end{array}\right\}$

*20. $\left.\begin{array}{l} x^3-3x^2+y=0 \\ y+12x=28 \end{array}\right\}$

*21. $\left.\begin{array}{l} x^3-y+1=0 \\ x^2+y^2-x-2y=0 \end{array}\right\}$

*22. $\left.\begin{array}{l} x^2y=4 \\ x^2-y^2+3=3x \end{array}\right\}$

CHAPTER VIII.

ON THE SIMPLEST GRAPHS PASSING THROUGH GIVEN POINTS.

54. When certain points have been plotted, it is often desirable to find what is the *equation* to the *simplest graph* that can be drawn through them.

When only *two* points are given, say, (a, b) and (c, d), they can be joined by *a straight line*. Suppose $y=px+q$ is the equation to this line (where p and q are some constant quantities which have still to be found). Then since the equation is to be satisfied by $x=a$, $y=b$, we have

$$b=pa+q;$$

similarly $$d=pc+q.$$

Solving these simultaneous equations* for p and q, we find $$p=\frac{b-d}{a-c},\ q=\frac{ad-bc}{a-c}.$$

Thus the equation to the graph is known, viz.—

$$y=\frac{b-d}{a-c}\cdot x+\frac{ad-bc}{a-c}.$$

55. **Graph through Three Points.**—When *three* points are given, say (a, b), (c, d) and (e, f), it will generally be impossible to draw a straight line through them all. Therefore in the equation to the *graph* joining them, y cannot have such a simple form as $px+q$; the form for y that would come next in simplicity is $y=rx^2+px+q$, containing in addition a term in x^2 (not in x^3 or x^4, etc., which are more complicated than x^2).

* These equations may be inconsistent, in which case the equation to the line is not of the form $y=px+q$, but of the only form not included in that viz. $x=p$. See Ch. II. § 14.

If this is the equation to the graph, then, since it is satisfied by $x=a$, $y=b$,

we have $\qquad b=ra^2+pa+q;$

similarly $\qquad d=rc^2+pc+q,$

and $\qquad f=re^2+pe+q.$

Thus there are three equations, of the first degree, for finding the three unknown constants p, q and r. These can easily be solved when a, b, c, etc., are given numerically, and then we have found the simplest function of x (viz. rx^2+px+q), the graph of which will pass through the three given points.

The method of solution shows that in any given case there cannot be *several* values found for p, q, r, but only *one* value for each, since the equations from which p, q, and r are derived are of the first degree.

Hence there is only *one* equation of the form $y=rx^2+px+q$ such that the graph corresponding to it will pass through the three given points. The graph is a parabola (§ 48).

Similarly, if y is a function of the *third* degree in x, the corresponding graph can be made to pass through *four* given points; and generally, if y is of the $(n-1)$th degree in x the corresponding graph can be made to pass through n given points.

Ex.—Find the equation to the simplest graph through the points $(0, -3)$, $(1, 0)$, $(2, 3)$, $(3, 6)$.

Assume as the equation to the graph (see end of § 55)

$$y=px^3+qx^2+rx+s.$$

Substituting successively the above pairs of values for x and y we have

$$-3=s \quad \text{(i)}$$
$$0=p+q+r+s \quad \text{(ii)}$$
$$3=8p+4q+2r+s \quad \text{(iii)}$$
$$6=27p+9q+3r+s \quad \text{(iv)}$$

Eliminating r between (ii) and (iii), and substituting for s from (i) we have

$$3p+q=0 \quad \text{(v)}$$

Eliminating r between (ii) and (iv), and substituting for s from (i) we have

$$4p+q=0 \quad \text{(vi)}$$

Solving (v) and (vi) we find $p=q=0$.
Whence and from (ii) $r=-s=3$.

Thus the equation to the required graph is

$$y=3x-3.$$

Whence the four given points lie on a straight line.

In other examples the values of p and q may not be 0, and we then get an equation of higher degree in x.

Examples VIII.

1. Find the equation to the simplest graph passing through the points given:

(i) $(2, 1), (1, -2)$. (ii) $(2, -1), (-2, 1)$. (iii) $(2, 1), (2, 2)$. (iv) $(1, 2), (2, 2)$.

2. Find the simplest function of x whose graph passes through the following points:

(i) $(0, -2), (1, 0), (2, 4)$. (ii) $(0, -3), (1, 2), \left(\frac{3}{2}, 0\right)$.

(iii) $(0, -2), \left(\frac{1}{2}, -\frac{11}{8}\right), \left(-1, -\frac{5}{2}\right)$.

(iv) $(-1, 2), \left(\frac{1}{3}, -\frac{4}{3}\right), \left(\frac{1}{2}, -1\right)$.

(v) $\left(0, \frac{1}{3}\right), \left(-\frac{2}{3}, 0\right), \left(\frac{2}{3}, \frac{2}{3}\right)$.

(vi) $\left(1, -\frac{5}{2}\right), (-2, -1), (4, -4)$.

3. Prove that the four points

$$(-2, 3), \left(-1, \frac{1}{2}\right), \left(1, -\frac{3}{2}\right), (2, -1)$$

are on a graph whose equation is of the form $y=ax^2+bx+c$, and draw the graph.

4. Prove that the four points

$$(1, -8\tfrac{1}{2}), (2, -5), (-2, -13), (3\ \ 4\tfrac{1}{2})$$

satisfy an equation of the form

$$y=ax^3+bx^2+cx+d$$

and draw the graph, plotting additional points from the equation.

ANSWERS

Examples I. (*Page* 5.)

4. 30·25, 31·55, 32·65. 8. 4·94.
9. Between stations 2 and 3.
10. Between stations 1 and 2.
11. 3 gall. 12. 3 gall. ; 3 gall. per min.

Examples II. (*Page* 20.)

6. (i) $y=x+3$. (ii) $y=-x+7$. (iii) $y=-\frac{x}{2}+3$. (iv) $y=-\frac{x}{10}-\frac{16}{5}$.

7. 16·14. 8. $15\frac{5}{9}$, $37\frac{7}{9}$, -40. 9. $B=\frac{9}{5}A+32$.

11. The reading 55·6 is too large. 12. 121, 102, 85.

15. 12·25, ·578, 2·65, ·894. 16. 324 ft. $1\frac{1}{4}$ secs.

17. 36·7 sq. ft., 6·67 ins. 18. 1·29 sq. ft.
19. 9·9 cub. ft.
21. $a=-5·16$, $b=·9$. £172 10*s*., £145 6*s*. 8*d*., £92 10*s*.

Examples III. (*Page* 28.)

1. (i) min. $-\frac{1}{4}$. (ii) min. $-\frac{1}{4}$. (iii) max. 0.

   (iv) max. $\frac{25}{12}$. (v) min. 0. (vi) max. $\frac{9}{16}$; min. -1, twice.

2. (i) min. -3, max. 1. (ii) min. 0, max. 1.

   (iii) min. 0, max. $\frac{4}{27}$. (iv) min. $-·385$, max. ·385.

   (v) min. -4, max. -8. (vi) min. 2, max. -6.

3. max. 25. 4. 14,400 sq yds. 5. 12 in.
6. 311·1 yds. 7. 50 sq. in. 8. 5·656 in.
9. 207·8. 10. 592592·6. 11. 96·22 cub. ft.

Examples IV. (*Page* 36.)

1. $-·25$. 2. 1·4. 3. -1, -2. 4. 6, -5.
5. ·5, -2. 6. ·7, -1. 7. 3. 8. $-·5$.
9. Imaginary. 10. Imaginary. 11. ·4, $-·6$. 12. 1·7, ·3.
13. 20, -15. 14. 17·2, 16·5.
15. (i) -1, -3. (ii) -2. (iii) ·5, $-4·5$.

16. (i) 1·7, $-1·2$. (ii) $\frac{1}{4}$. (iii) 1·5, -1. 17. 1.

18. -1. 19. 1·6, $-·6$. 20. 1, -2. 21. -1.
22. -1, -2. -3. 23. 1, $-·5$. 24. ·5, -1.

EXAMPLES V. (*Page* 45.)

1. $-1, 1$ 2. $-1, 3$. 3. $1, \infty$. 4. $1, 0$. 5. $0, 1$.
6. $-\infty, 1$. 7. 0. 8. ∞. 9. $-\infty$. 10. 0.
11. 1. 12. ∞. 13. -1. 14. -3.

EXAMPLES VI. (*Page* 54.)

1. Symmetrical about axis of y. 2. Symmetrical about both axes.
3. Non-symmetrical. 4. Symmetrical about axis of x.
5. Symmetrical in opposite quadrants.
6. Symmetrical about axis of x. 7. Circle.
8. Rectangular Hyperbola. 9. Ellipse. 10. Hyperbola.
11. Circle. 12. Circle. 13. Parabola.
14. Parabola. 15. Two straight lines.
16. Two straight lines. 17. Ellipse. 18. Ellipse.
25. Hyperbola. 26. Hyperbola.

EXAMPLES VII. (*Page* 59.)

(Most answers are given only to one decimal place.)

1. $x=1, y=2$. 2. $x=5, y=4$. 3. $x=2, y=3$.
4. $x=5, y=3$. 5. $x=1\cdot6, y=1$. 6. $x=8\cdot8, y=-1\cdot2$.
7. No solution. 8. No solution.
9. $x=\cdot6$ or $-1\cdot6$, $y=1\cdot4$ or $3\cdot6$. 10. $x=0$ or $-\cdot5$, $y=\cdot5$ or $\cdot4$.
11. $x=2\cdot7$ or $-\cdot7$, $y=\cdot7$ or $-2\cdot7$. 12. $x=-2, y=1$.
13. $x=4\cdot8$ or $1\cdot9$, $y=-4\cdot6$ or $1\cdot2$. 14. $x=2\cdot1, y=\cdot37$.
15. $x=\pm3\cdot6$ or $\pm1\cdot6$, $y=\pm1\cdot6$ or $\pm3\cdot6$.
16. $x=5\cdot5$ or $\cdot85$, $y=-\cdot55$ or $-3\cdot4$.
17. $x=1$ or -2, $y=\cdot5$ or 5. 18. $x=-15, y=-3$.
19. $x=0, 2\cdot5$, or $-12\cdot5$, $y=0, 1$ or -5.
20. $x=2, 4\cdot3$ or $-3\cdot28$, $y=4, -23\cdot2$ or $67\cdot4$.
21. $x=1$ or $-\cdot58$, $y=2$ or $\cdot81$.
22. $x=2$ or $-1\cdot2$, $y=1$ or $2\cdot8$.

EXAMPLES VIII. (*Page* 62.)

1. (i) $y=3x-5$. (ii) $y=-\frac{x}{2}$. (iii) $x=2$. (iv) $y=2$.

2. (i) x^2+x-2. (ii) $2x^2-x-3$. (iii) $\frac{1}{2}x^2+x-2$.

(iv) $3x^2-\frac{1}{2}x-\frac{3}{2}$. (v) $\frac{1}{2}x+\frac{1}{3}$. (vi) $-\frac{1}{2}x-2$.

3. $a=\frac{1}{2}, b=-1, c=-1$.

4. $a=\frac{1}{2}, b=0, c=0, d=-9$.

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